Back to Start

Linking Geometry, Algebra and Calculus with GeoGebra

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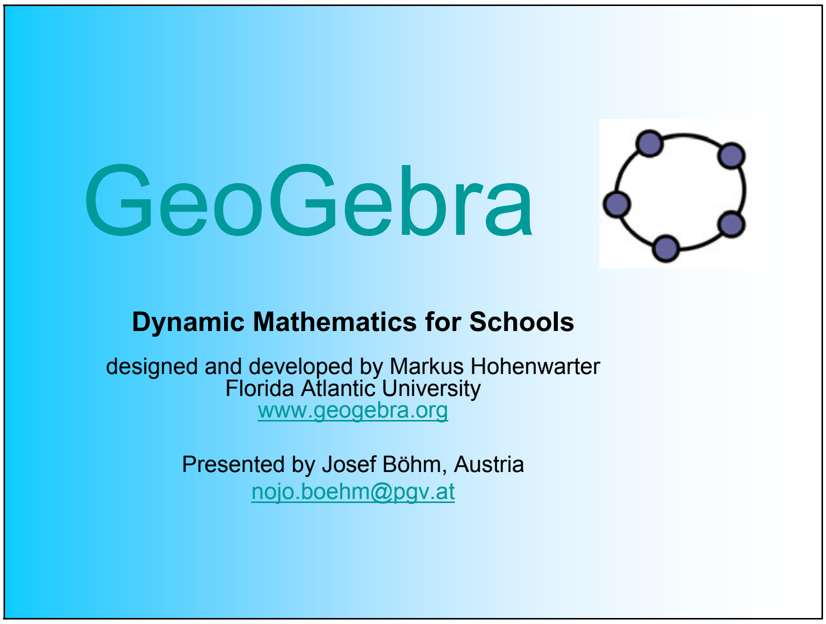
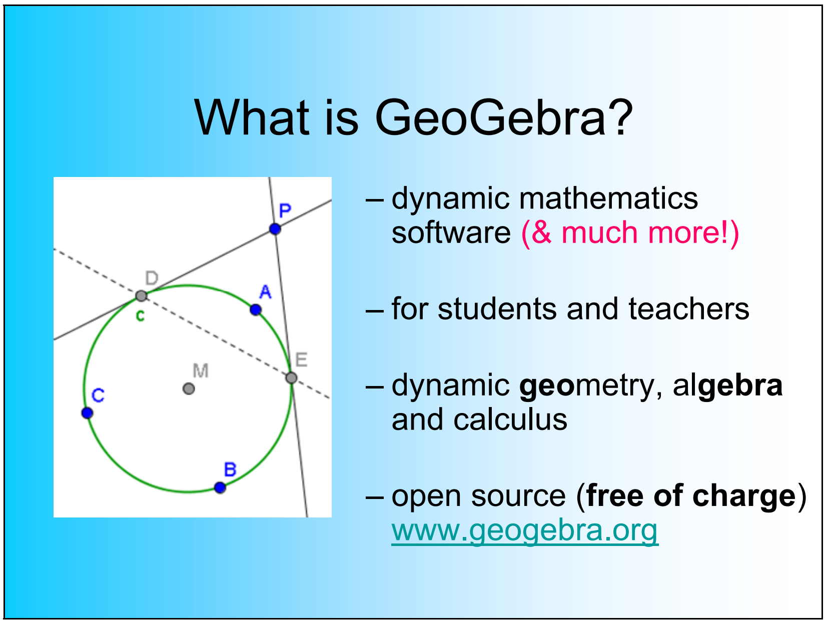
Abstract
GeoGebra is free, open-source, and multi-platform software that combines dynamic geometry, algebra and calculus in one easy-to-use package. Students from middle-school to university can use it in classrooms and at home. In this presentation, we will introduce the features of GeoGebra and outline plans for developing training and research networks connected to GeoGebra.

I was asked by my friend Markus Hohenwarter to present the recent version of GeoGebra at TIME 2008. I did not hesitate because I am an enthusiastic user of GeoGebra and I am happy to be able to demonstrate the power of this program to a high classed auditorium.

Comment to the paper:
Dynamic Geometry needs to be presented dynamically which is impossible in a written paper. I include some screenshots of the demonstration. All GeoGebra files (extension .ggb) are provided in the folder TIME08\Boehm\TIME GeoGebra. I recommend to experiment with the files. Simply go to the website www.geogebra.org and start GeoGebra by using the webstart.

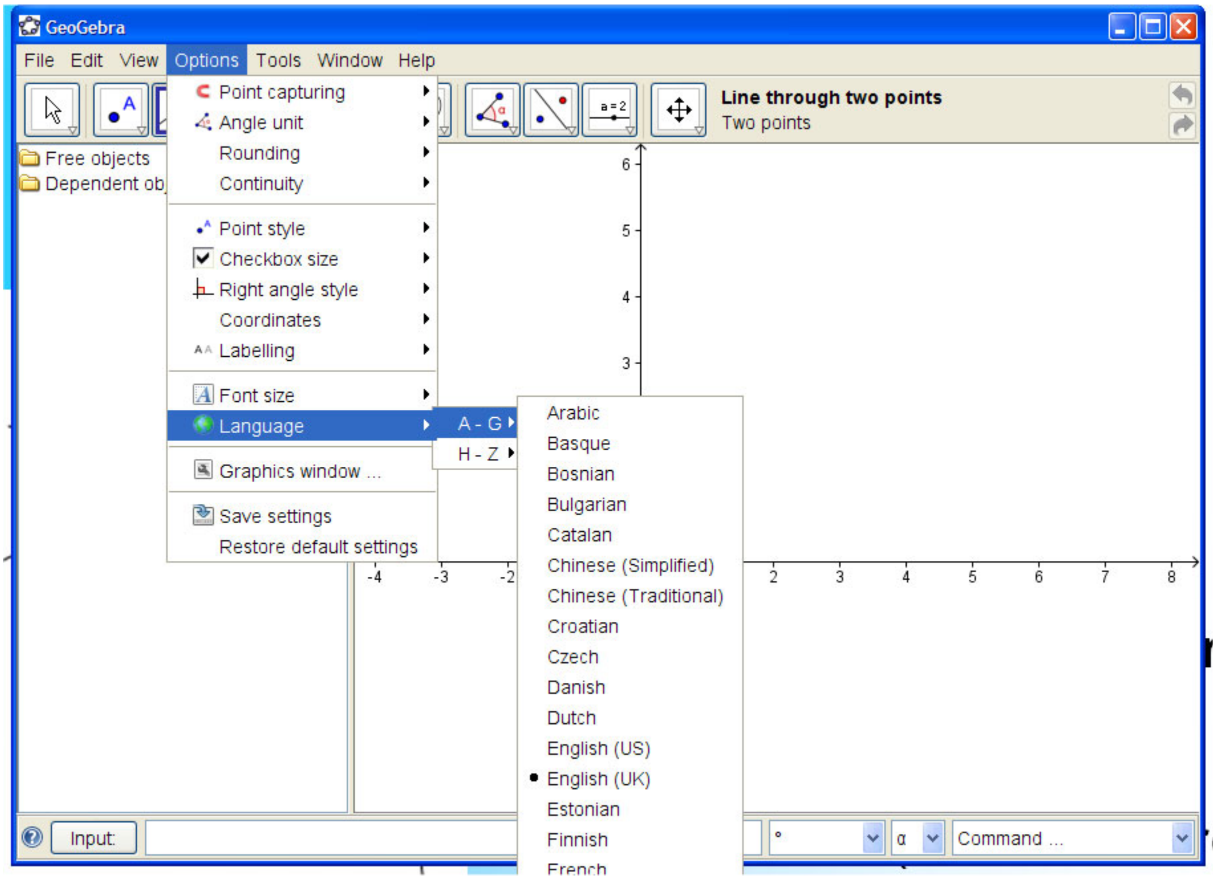
(You can download the program, too – it is open source and free of charge.)



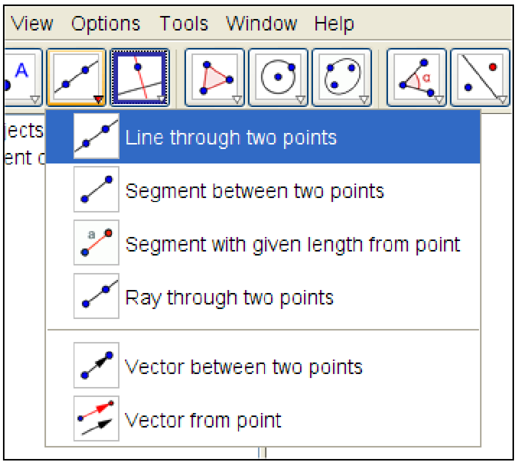


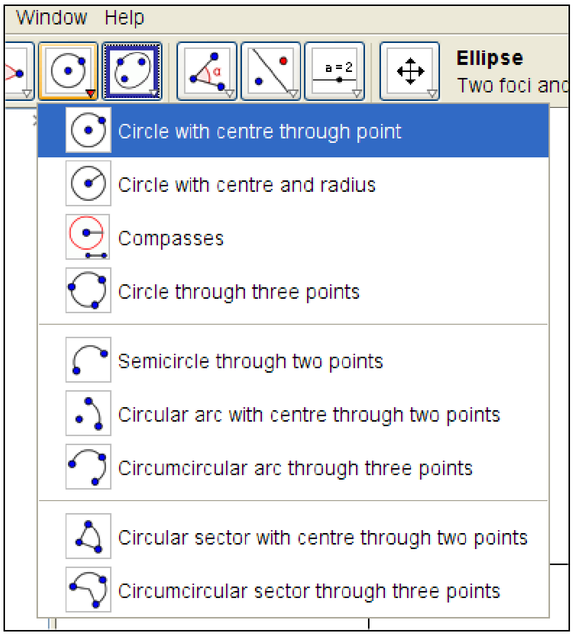
Opening GeoGebra you are presented with a Dynamic Geometry program which can be used very intuitively. We know this from numerous classroom experiences. Students don't need extended explications and introductory units. In the meanwhile GeoGebra is provided in more than 30 languages.

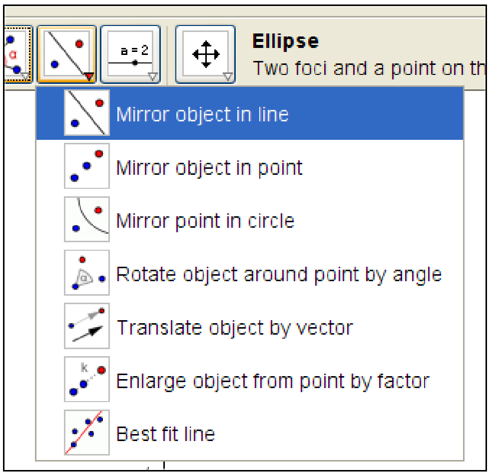
I show some menus to give an impression of the many features. The toolbar is customizable, own tools – produced by macros – can be added as buttons in the toolbar.

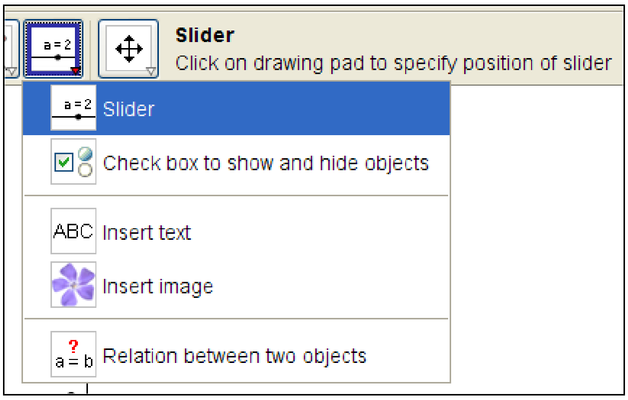


Some menus:







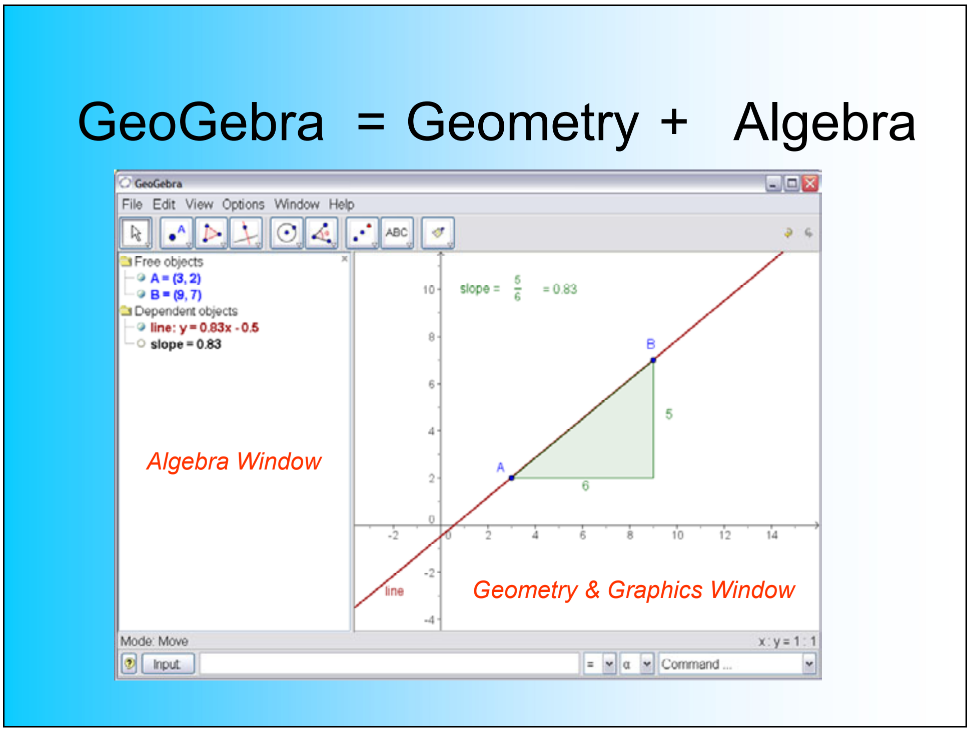


You can find all the activities which you are used to have in a dynamic geometry program but there are a lot of additional features which I'll demonstrate – at least partially – in the next 30 minutes.

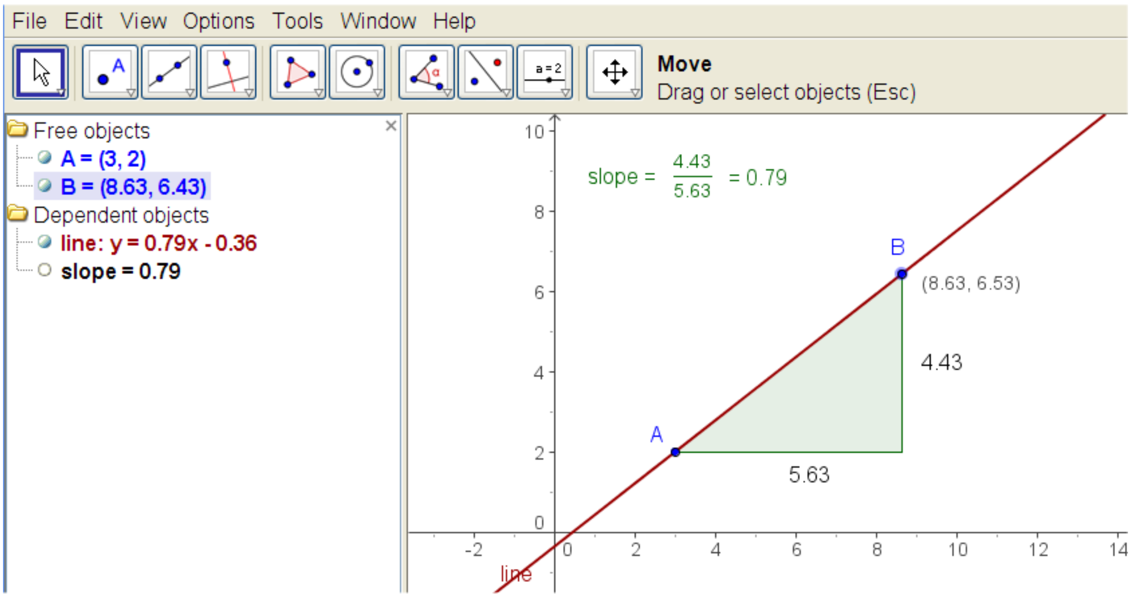
Why the name GeoGebra?

It is a combination of Geometry and Algebra – that's not too difficult to guess – but there is also a lot of Calculus included.

(Maybe that some Computer Algebra is hidden?)



file: slope.ggb



This file can be used to let the students experience the connection between the coordinates of two points and the slope of the line of the points.

You can see that we have a double representation: the geometric one and the algebraic one (=GeoGebra). One can enter the objects either as geometric objects (menus) or as algebraic objects – pairs of coordinates, functions – via the entry line. Moving the objects in the Geometry window changes the expressions in the Algebra window accordingly. Editing the expressions in the Algebra Window results in the respective change in the Geometry Window.

This is a main feature of GeoGebra meeting the demands of many didactics and educators to provide as many representations forms as possible for the students.

Bi(Tri)directional Combination

- Bidirectional change of representation

  symbolic ↔ visual (↔ numerical)
- Examples
  - coordinates ↔ point in coordinate system
  - circle's equation ↔ circle drawn in coordinate system
  - function's equation ↔ graph of function

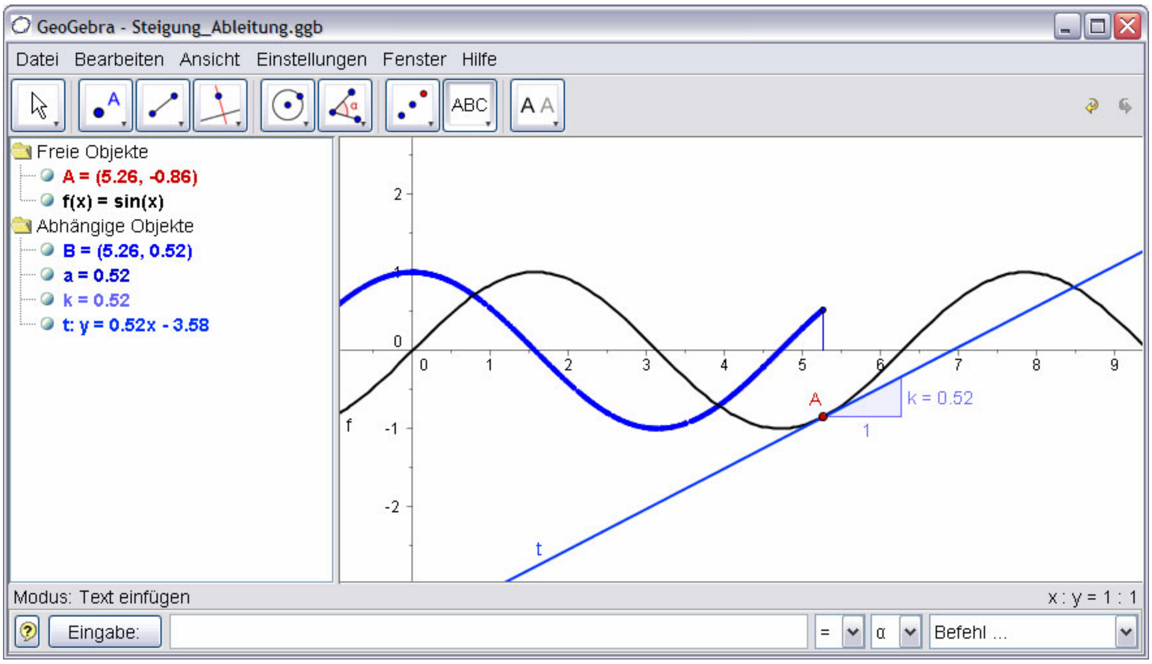
Now let's go to the demonstration examples:

Dynamic Calculus

- functions in x
- derivatives
- integrals

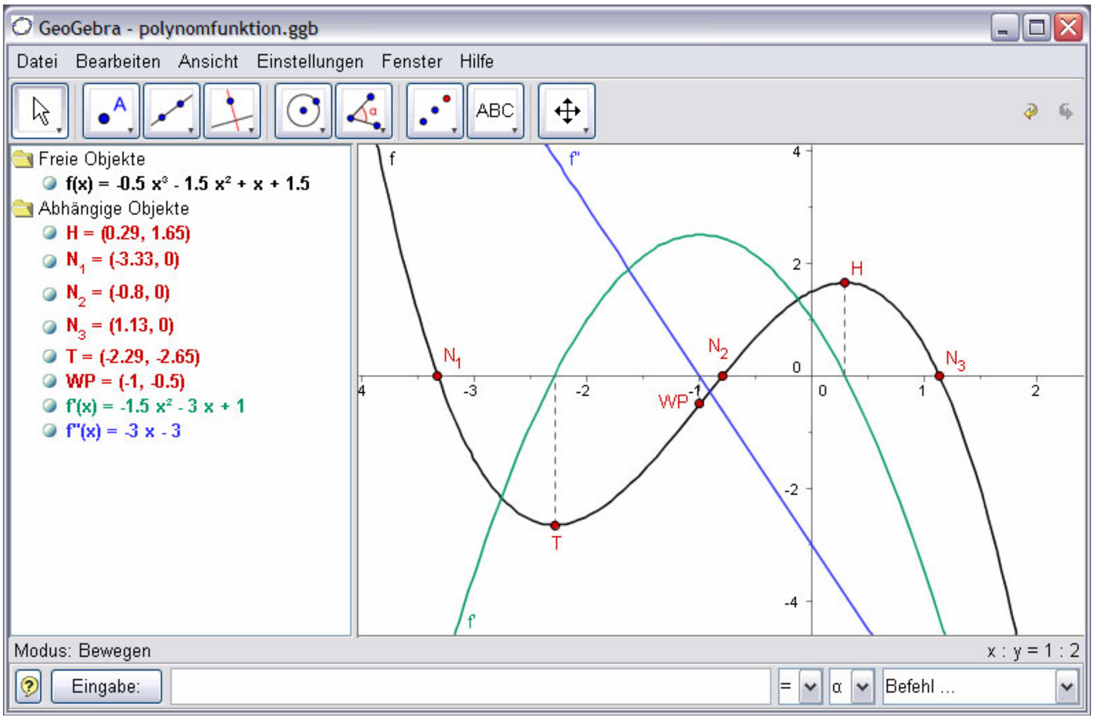
We can enter any function and show a visualization of generating the first derivative. Change f(x) in the algebra window and have other functions.

The fine thing is that the construction is so easy to do that it can be done together with the students.



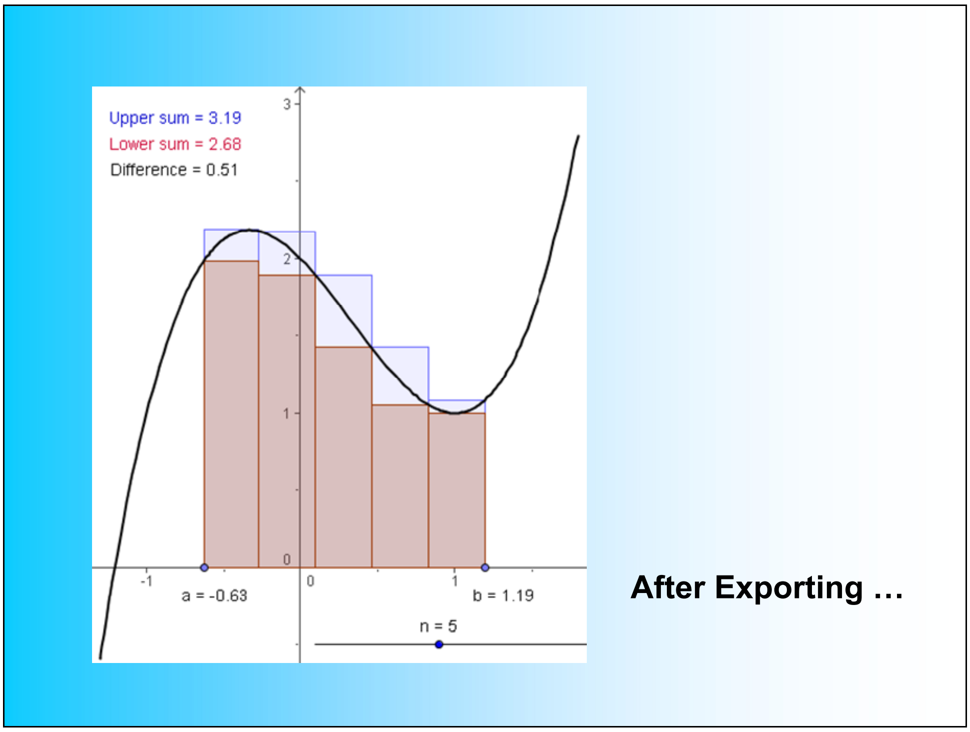
file deriv_sin.ggb

This screen shows a demonstration of the connection between a function graph and the graphs of the first and second derivative.



file function.ggb (provided by W. Wegscheider)

If we do something with derivatives then we should do something with integrals, too:



file: calc_upper_lower_sum.ggb

You can see the slider bar, which is used to change the number of rectangles.

The screen shows “After Exporting …”.

One of the great advantages of GeoGebra is that everybody is able to convert a GeoGebra file into an interactive website – and you need not to be an expert in website designing.

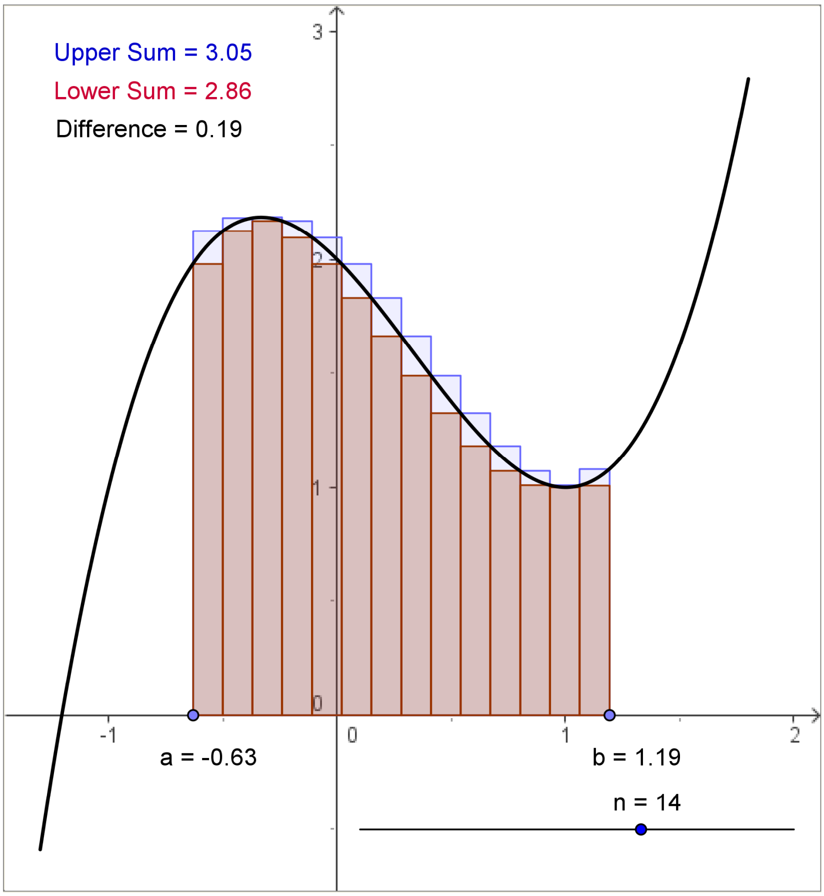
This website can be given to the students together with tasks to be performed and questions to be answered …

The next page shows the respective html document.

It can be found in the folder as LowerUpperSum.html.

Lower and Upper Sum of a Function

You can see the graph of a function together with its Lower Sum (red) and Upper Sum (blue) for an intervall [a, b].



1. Drag the boundaries a and b using the mouse. Which properties of the rectangles of the (a) Lower Sum, (b) Upper Sum can you observe?

2. Change the number n of rectangles by moving the respective point on the slider bar. What is the influence of n for Lower and Upper Sum?

3. Can your conjecture hold for other intervals [a, b]? Double Check your conjecture by changing the boundaries amd the number of rectangles.

Designed with GeoGebra by Markus Hohenwarter (transl. Josef Böhm)

I´ll continue with a collection of some examples which shall show the wide range of possibilities how to use GeoGebra. In fact I would need much more time to show all features.

- Pedal Curves x 2

Some Examples which are not expected within a DGS

- Multiplying fractions
- Interactive html products
- Background pictures passive & active
- sin(sin(sin(sin …)
- Probability Trees
- Taylor Expansion

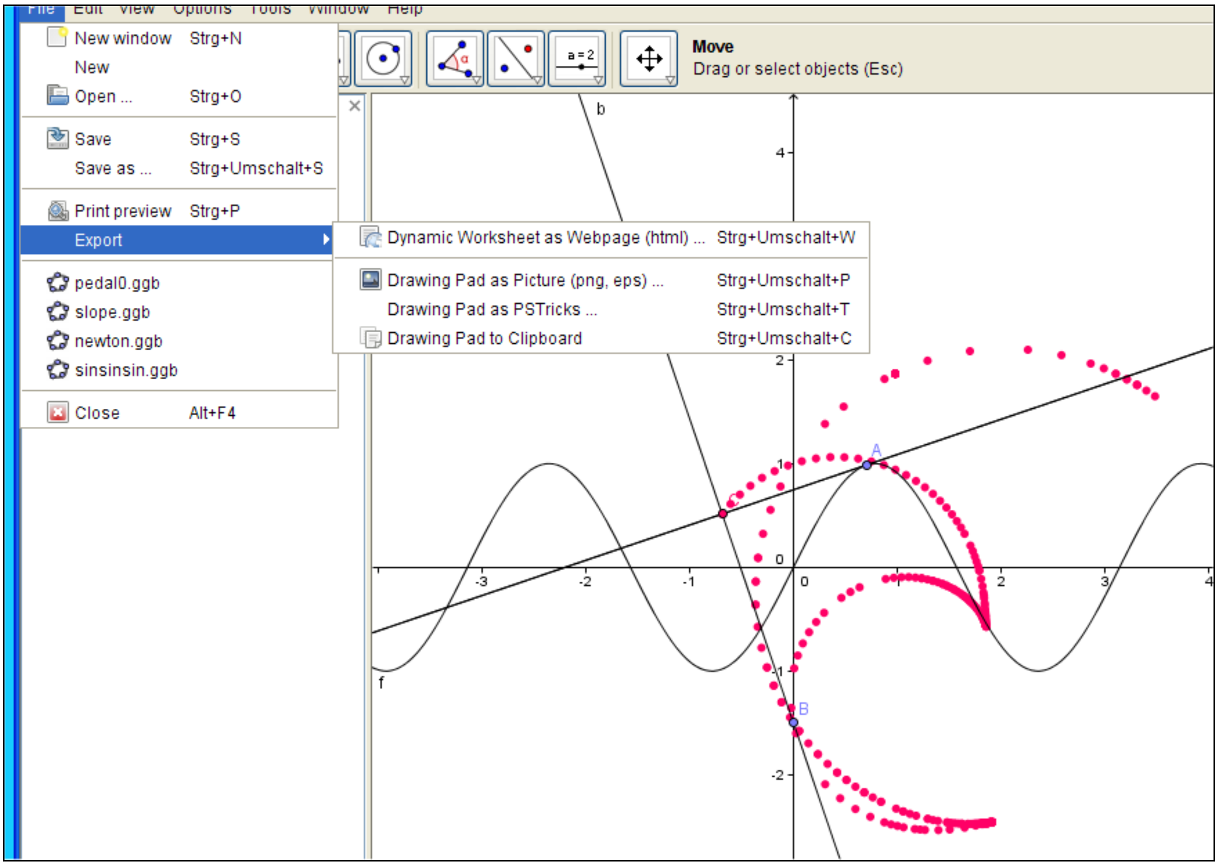
Some time ago I found a nice article on **Pedal Curves**. I treated this class of curves using DERIVE and the TI-software family and then I changed to dynamic geometry.

> Take any curve *f*. Fix one point in the plane *P* – the pole. Then take the family of all tangents of the given curve and draw the perpendicular lines through the pole wrt to the tangents. The locus of the intersection points tangent-perpendicular line is the pedal curve of *f* wrt to *P*.

We can treat this problem in several ways.

First way:

I use GeoGebra as a Dynamic Geometry program. Tangents and Perpendicular lines are taken from the menus as Black Boxes. We can trace the intersection point and finally let draw the locus.



file: pedal0.ggb

Export shows how to export the construction among others as a webpage!

Second way:

I used a CAS (DERIVE) to perform the necessary calculations which resulted in the parameter form of the curve. The pole is not fixed but moving on the *y*-axis P(0|*a*).

$$\mathrm{pedalcurve}\left(\left[t,\ \frac{t^3 - 4\cdot t}{5}\right],\ [0,\ a]\right)$$

$$\left[\frac{(3\cdot t^2 - 4)\cdot(5\cdot a + 2\cdot t^3)}{9\cdot t^4 - 24\cdot t^2 + 41},\ \frac{a\cdot(3\cdot t^2 - 4)^2 - 10\cdot t^3}{9\cdot t^4 - 24\cdot t^2 + 41}\right]$$

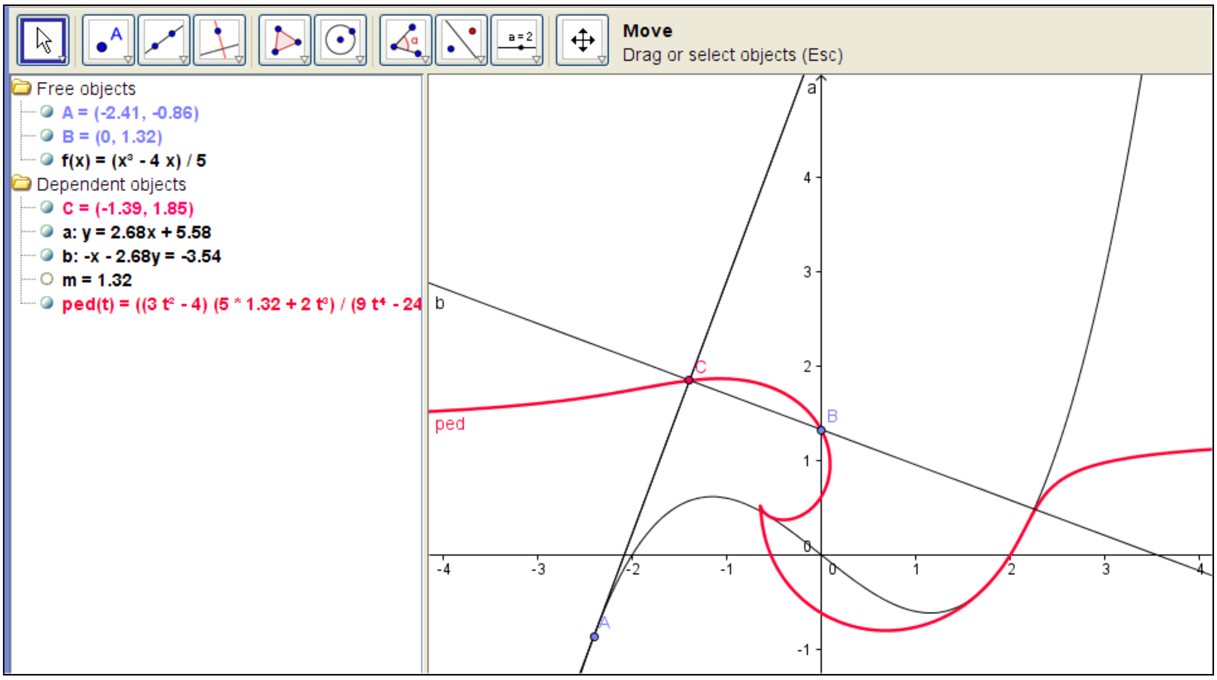
TimesOperator := Asterisk

$$\left[\frac{(3*t^2 - 4)*(5*a + 2*t^3)}{9*t^4 - 24*t^2 + 41},\ \frac{a*(3*t^2 - 4)^2 - 10*t^3}{9*t^4 - 24*t^2 + 41}\right]$$

The expression for the curve can be brought to GeoGebra by copy and paste.

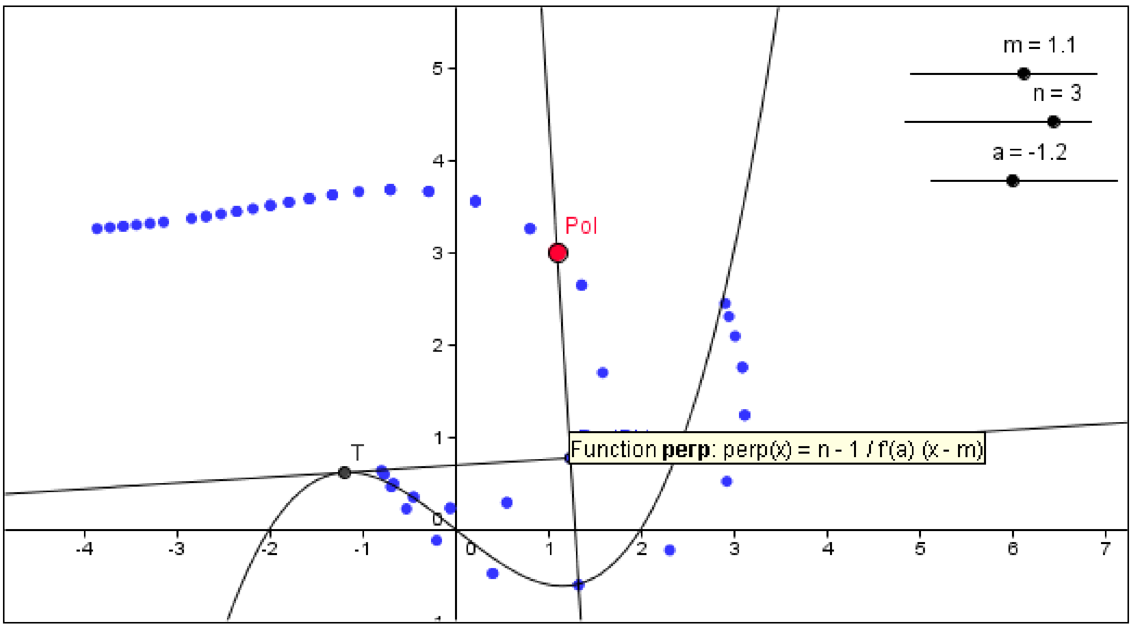
This is the result:

Moving A along the curve leads point C along the precalculated graph. Moving B along the y-axis changes accordingly the expression *ped*(t) and its graphic representation.



file: pedal1.ggb

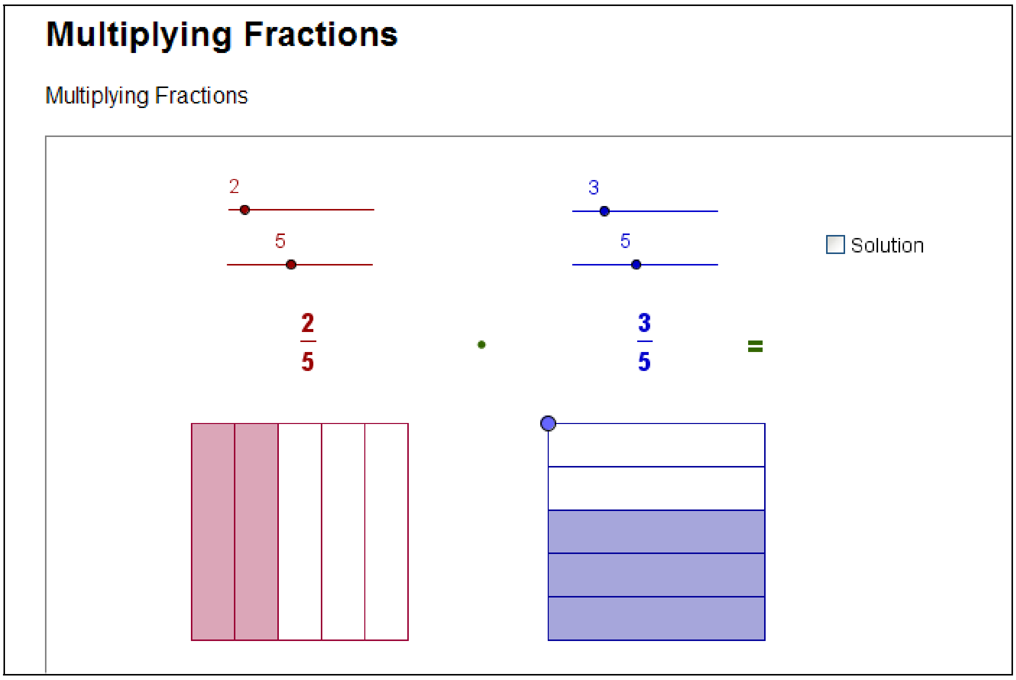
In my third version I don't use the Black Boxes Tangent and Perpendicular line, but enter the equation of the tangent and the perpendicular using common calculus notation. Additionally I keep the pole variable P(m|n). a is the x-value of point T. The slider bars do all the work. Finally I could add the generalized parameter form for the pedal curve.



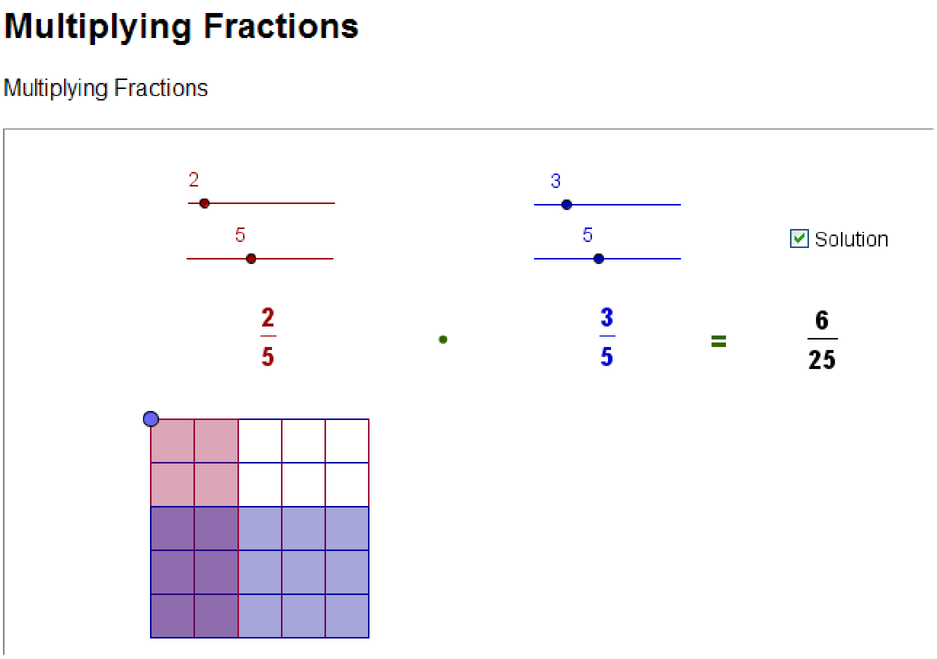
file: pedal2.ggb

A colleague from Bavaria uses GeoGebra in a very early stage: For visualizing Multiplying Fractions:

See also his work sheet as an html-document.



The slider bars define two fractions. Then move the right square by dragging it to the left over the left one. Interpret the solution, mark the little "Solution square" and compare!



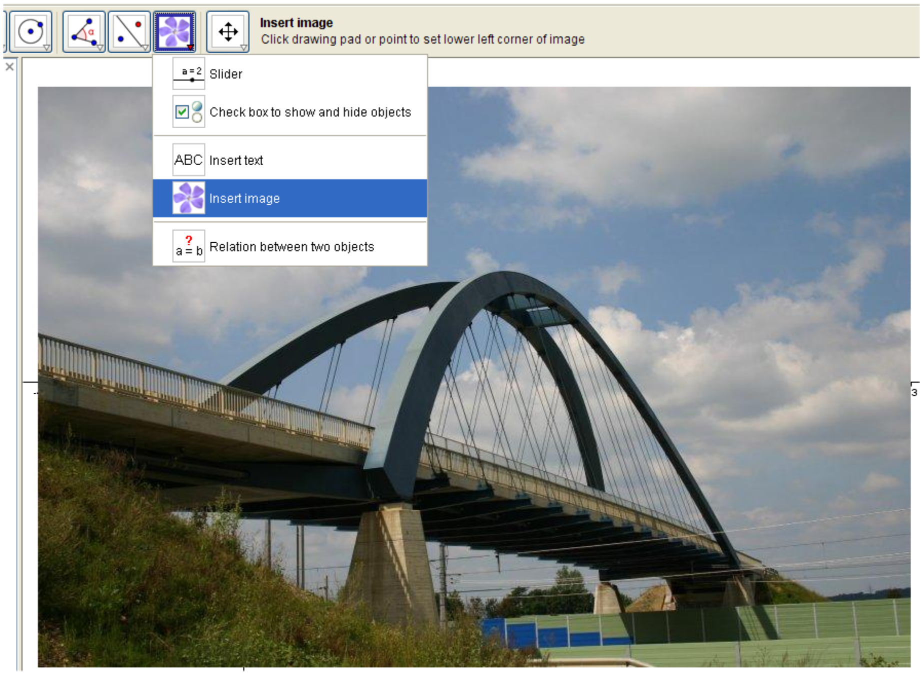
file: multiplying_fractions.ggb

Unfortunately it is not so easy to demonstrate sum, difference and quotient of fractions.

I announced "**Background Pictures passive and active**".

I would call the next application "*passive*". I load the picture (any graphic format) of a railway bridge close to my home in the background of the geometry window – more or less opaque.

The question is: What is the form of the bridge?
You might guess the answer of the students? "It is a parabola!".



Now let's check this using GeoGebra´s "5-point conic tool":

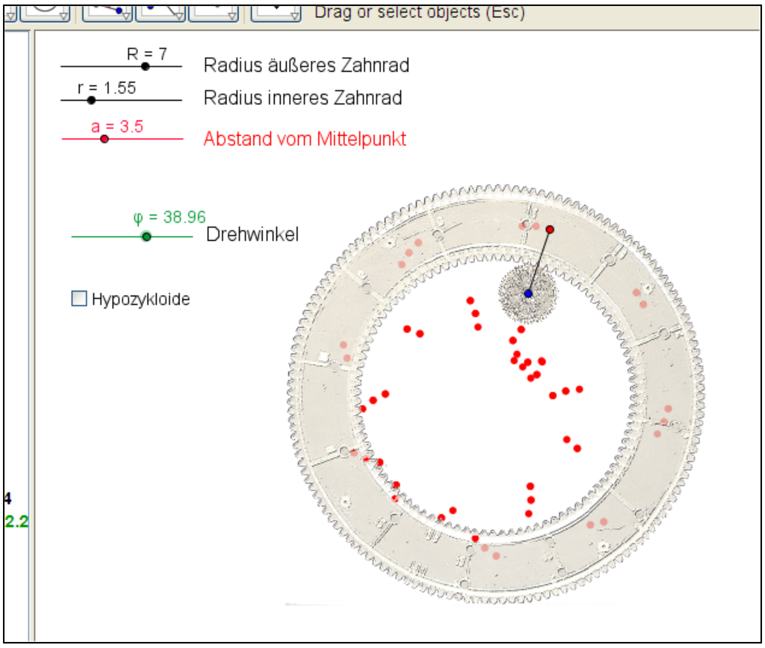
It is a hyperbola!

Next question: Is it really a hyperbola or does it only appear as a hyperbola. Could it be that in reality it is a parabola, or could it even be part of an ellipse???? What is your opinion?

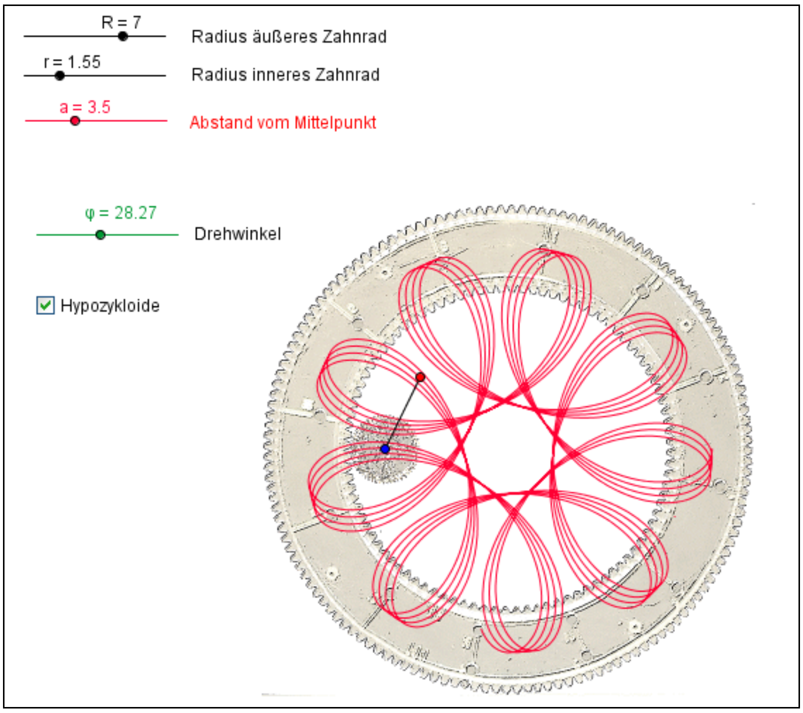
Next step would be a task for the CAS: how to find out the equation and the kind of the conic?

Now see an "**active**" background picture in GeoGebra (provided by Andreas Lindner, a teacher at the Gymnasium Ischl and head of the newly founded Austrian GeoGebra Institute).

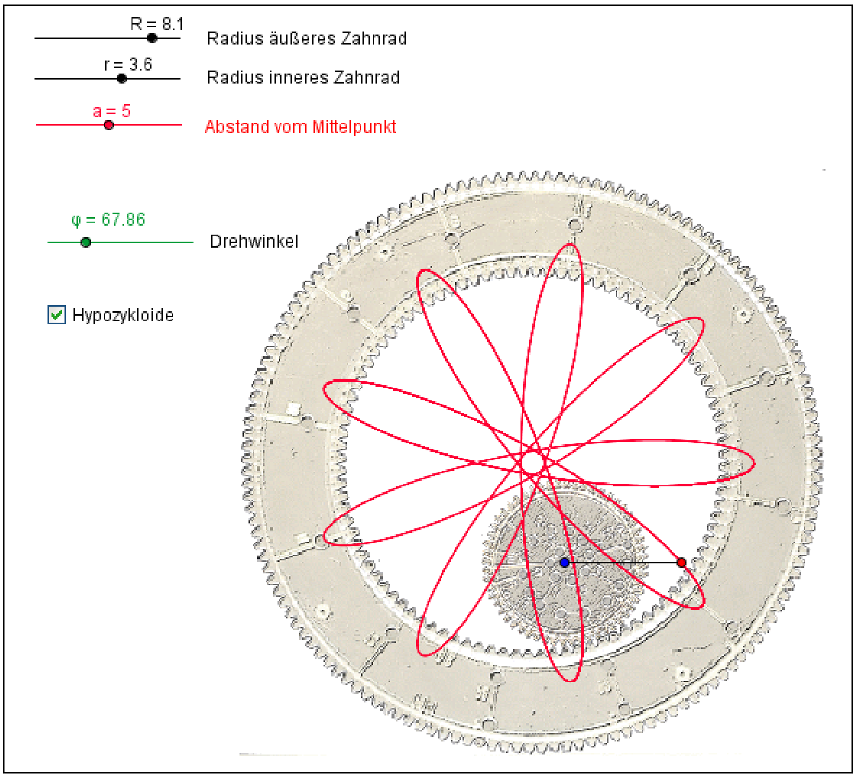
Andreas imports the pictures of two tooth wheels and connects the inner wheel with a GeoGebra object to produce "real" trochoids. Moving the green slider bar increases and decreases the rotation angle and lets the inner circle rotate. The trace of the red point describes the trochoid. *R* and *r* are the radii of the circles and *a* is the distance of the moving point from the centre of the inner circle.



file: spirograph.ggb

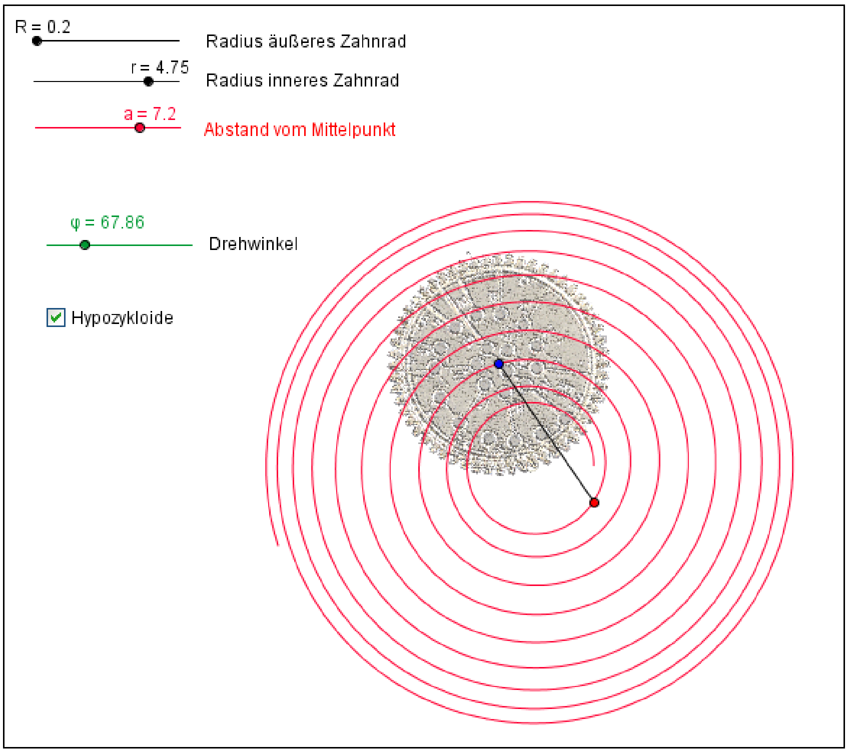


Marking the check square for "Hypozykloide" shows the locus of the point.



As you can imagine this is a wonderful "toy" for experimenting and playing around.

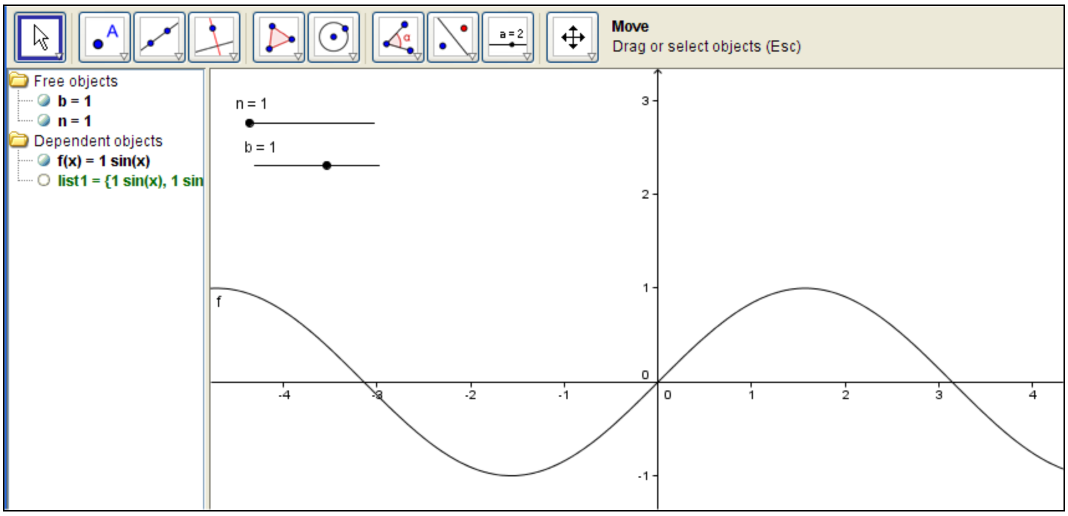
But not only this ….



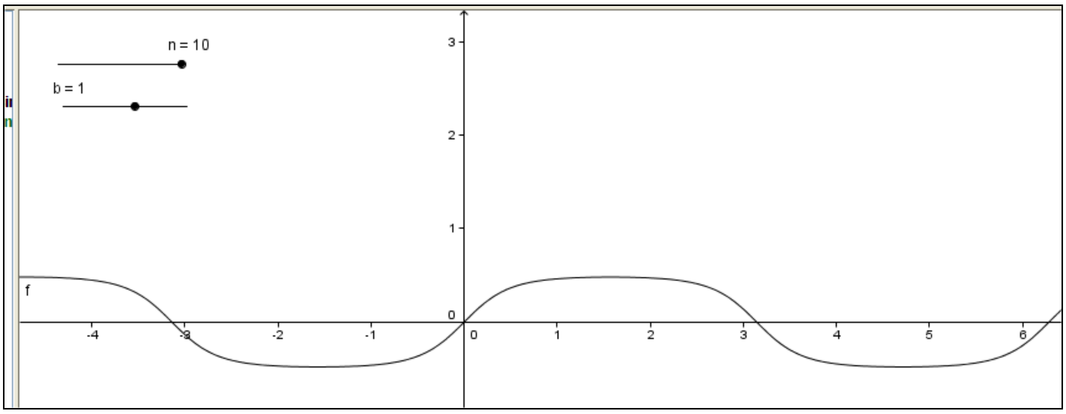
Can you explain the figure above?

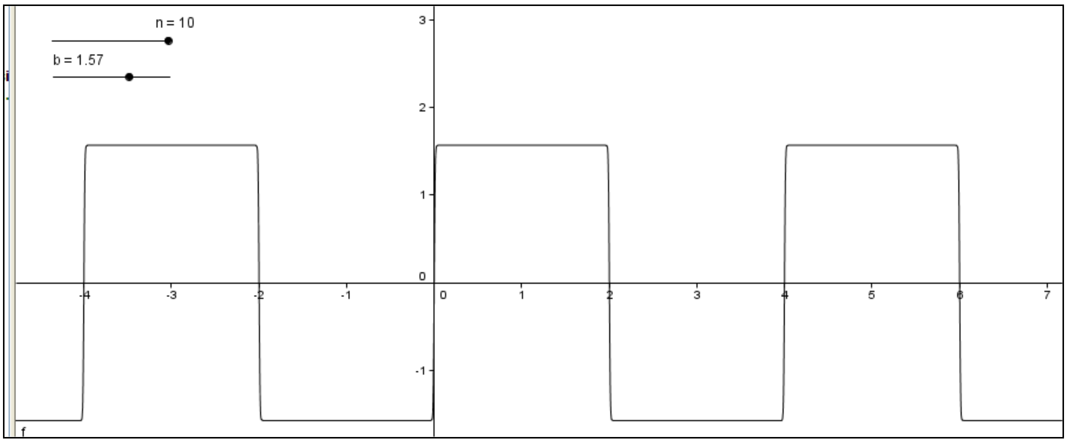
sinsinsin was presented by Hans Georg Weigand, Germany at the occasion of starting the Austrian GeoGebra Institute. He visualized the iteration of the sine-function.

His question to the audience was: What is the limit of the iteration of $\sin(x)$? The slider bar counts the number of iterations. Then he asked for the limit of the iteration of $\frac{\pi}{2}\cdot\sin(x)$?

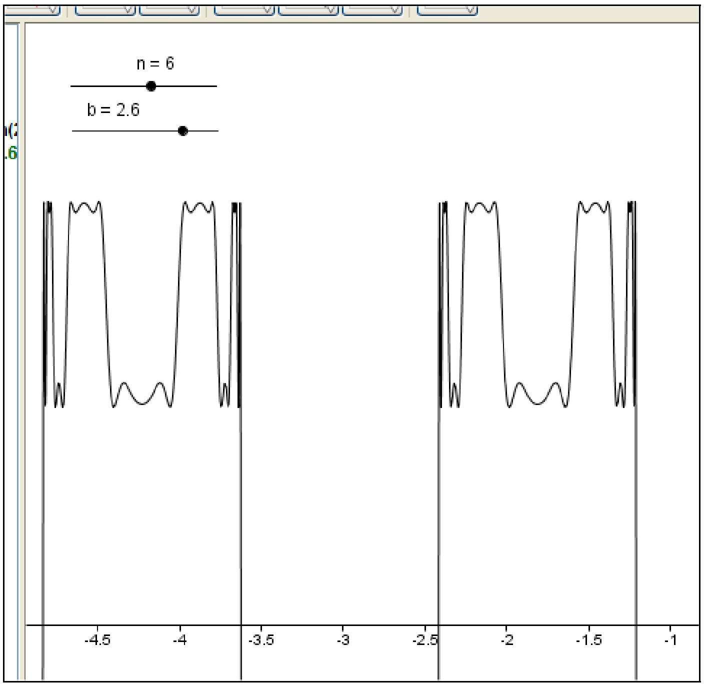


file: sinsinsin.ggb



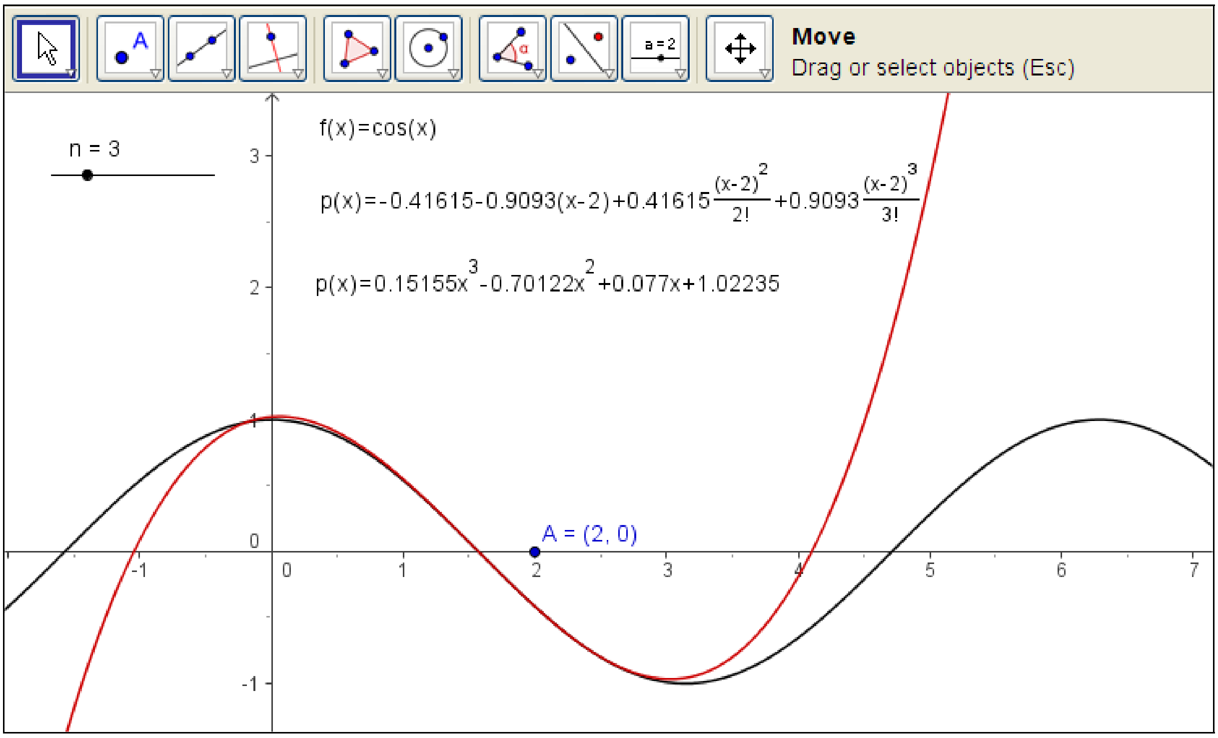


Hans-Georg had a slider bar for n and I added another one for b and tried values $> \frac{\pi}{2}$. What will happen now by increasing the number of iterations?



Strange figures are appearing!

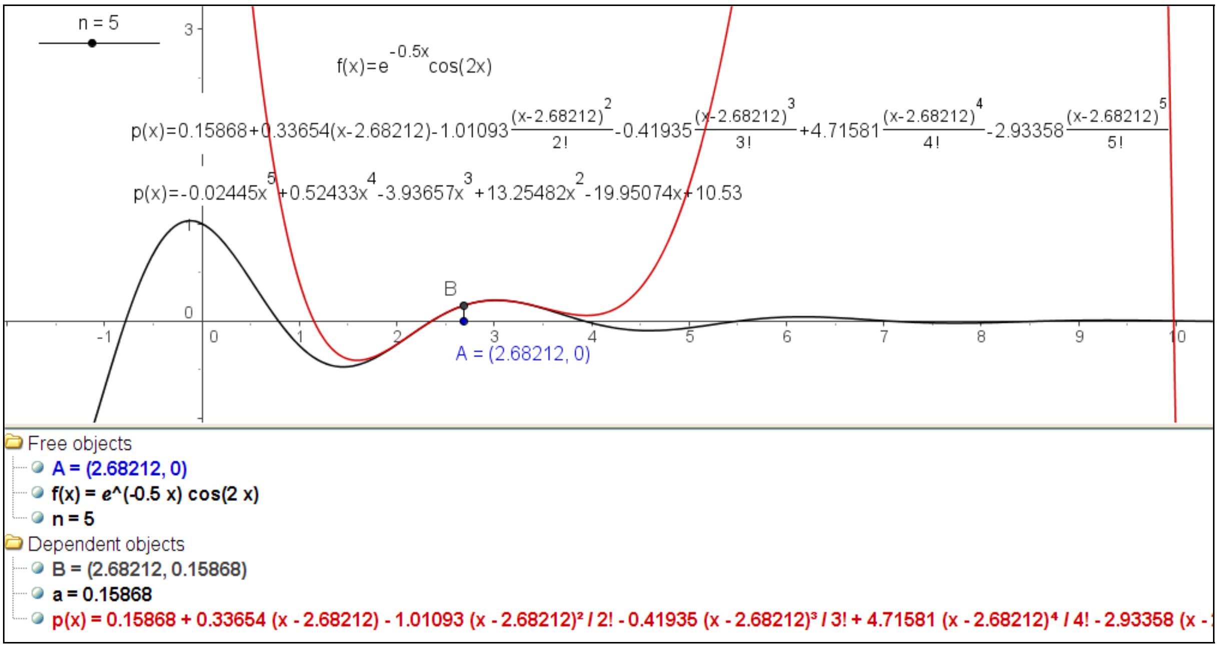
I will finish this section with a GeoGebra application which seems to be very close to Computer Algebra: visualization of Taylor Polynomials.

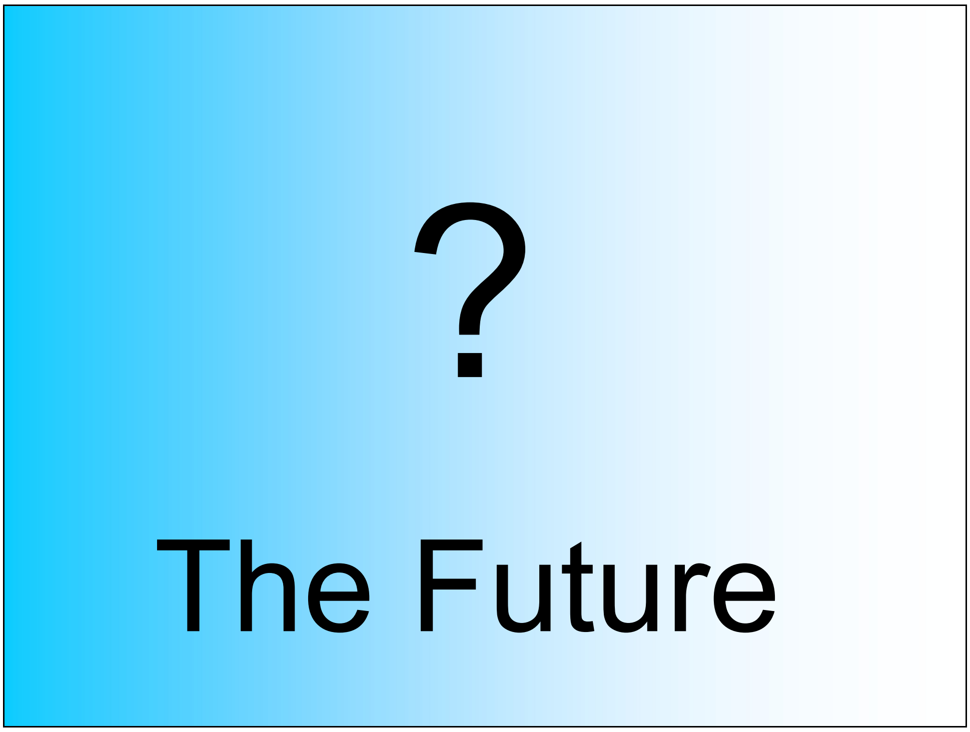


file: Taylorpolynom.ggb

You can move point A to place the position to develop the Taylor series and you can change the order *n* by using the slider bar.

This series is a little bit more complex. I split the screen. The only thing to enter is the function term for *f*(*x*). All the rest is done by GeoGebra.





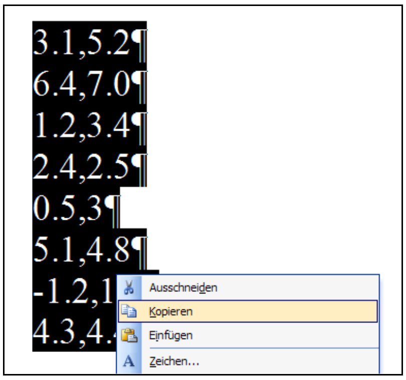
What is planned for the future?

Markus Hohenwarter and a few specialists are improving GeoGebra continuously step by step. Planned is implementation of a spreadsheet, of a computer algebra system and a 3D extension of the dynamic geometry.

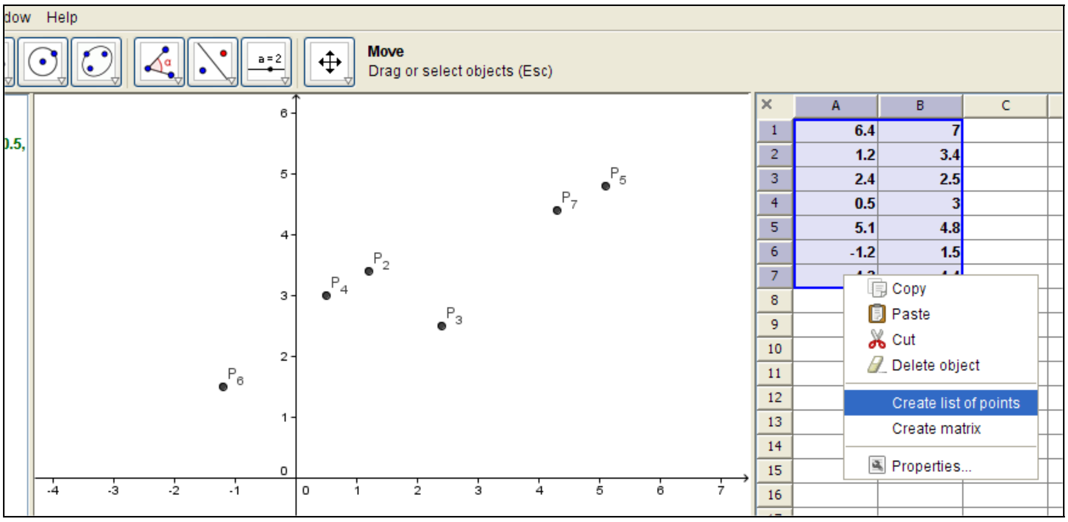
What I can show you are some impressions of a pre-pre-release of the spreadsheet within GeoGebra:

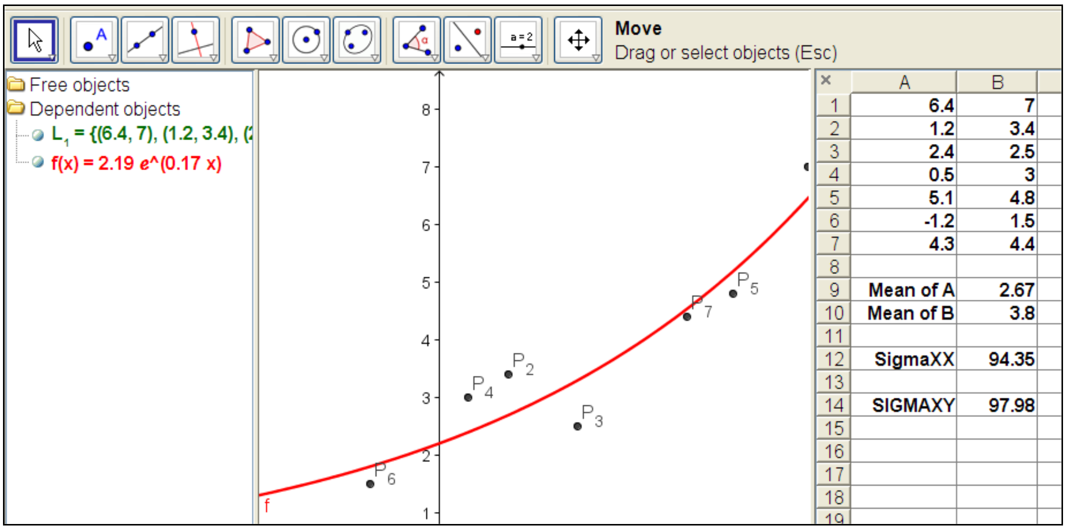
This is a CSV (comma separated table) of numbers in a WORD document.

I mark and copy (Ctr+C) the table.



I load GeoGebra and open the spreadsheet window. I can paste the table into the spreadsheet and represent the table in the algebra window as a List L_1 and in the geometry window as a set of points.

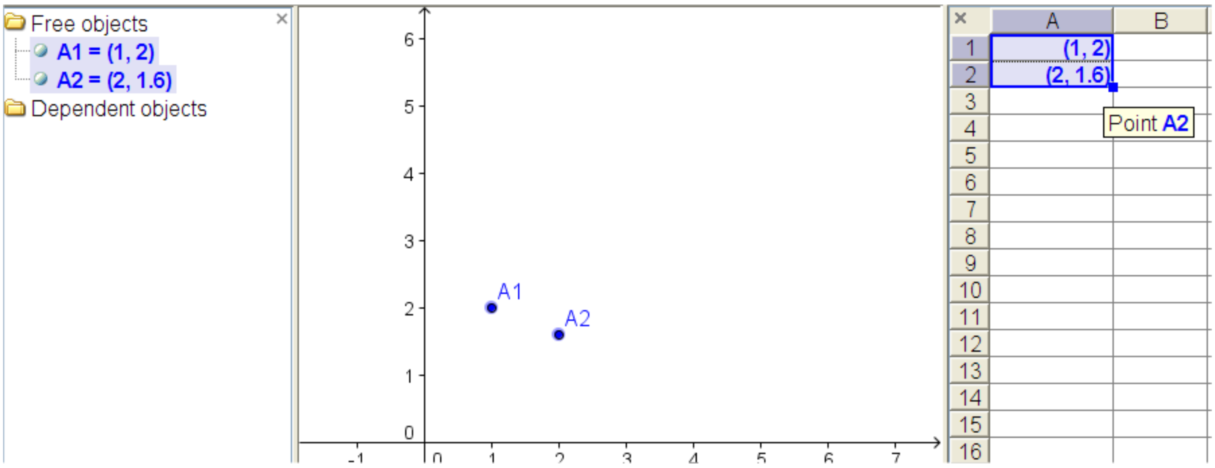




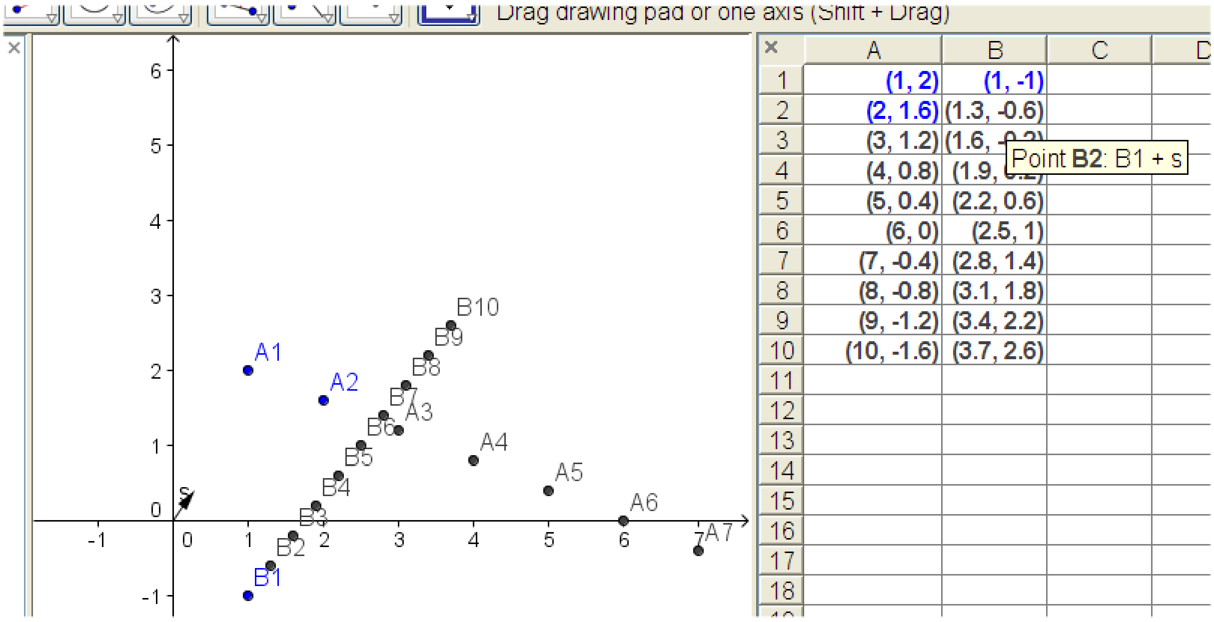
I can perform an appropriate regression (FITEXP(L_1), or any other) and I am able to do some statistics in the spreadsheet.

I enter two pairs of coordinates in cells A1 and A2, which immediately have their corresponding appearance in the algebra and geometry window.

You see that I highlight the two cells.



In Excel I can create a sequence of numbers, in GeoGebra I can create a sequence of points by dragging the little square (bottom right) down the column.



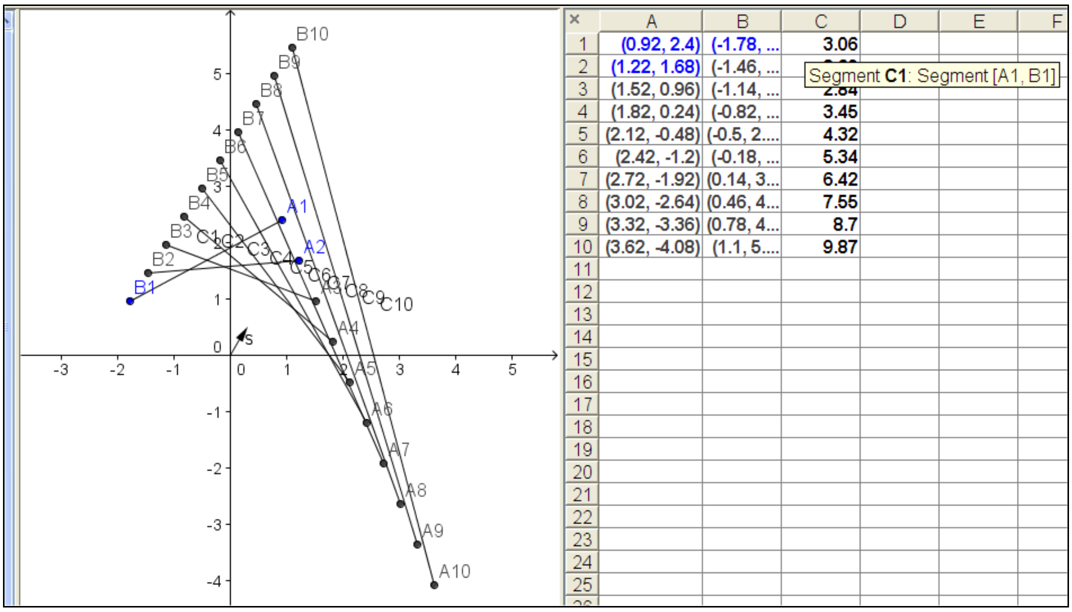
I create a second list in another way. I enter B1 = (1,-1) and the vector s = [(0,0),(0.3,0.4)] via the entry line.

You see my definition of point B2. Then I copy cell B2 down the column.

We have created two sequences of points.

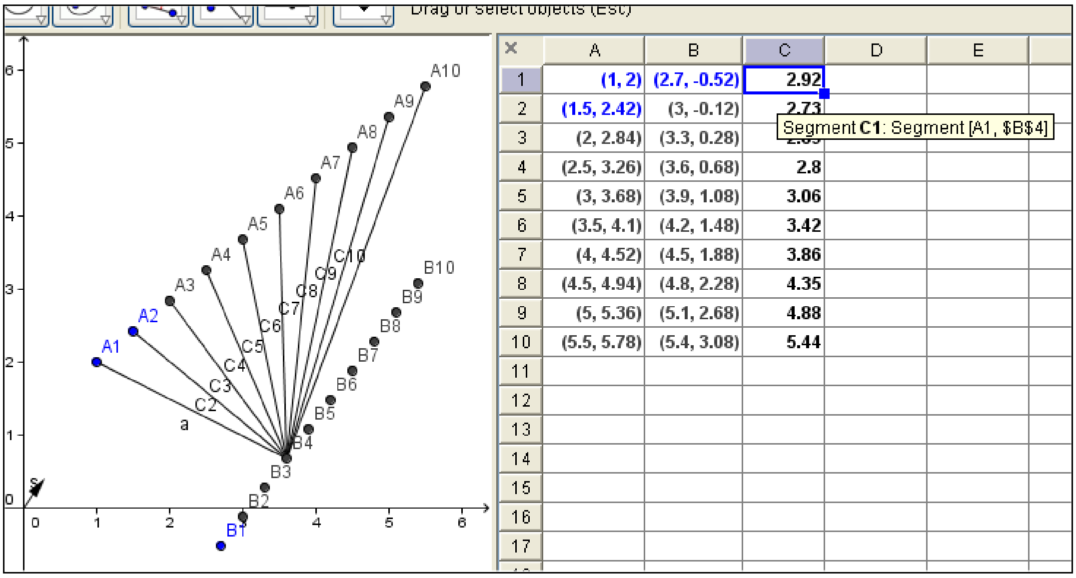
I proceed generating a sequence of segments first using relative references by defining C1=Segment[A1,B1] and copying down this formula.

The next screen shows applying the absolute reference by defining C1=Segment[A1,B4].



	A	B	C
1	(0.92, 2.4)	(-1.78, ...	3.06
2	(1.22, 1.68)	(-1.46, ...	[illegible]
3	(1.52, 0.96)	(-1.14, ...	2.84
4	(1.82, 0.24)	(-0.82, ...	3.45
5	(2.12, -0.48)	(-0.5, 2....	4.32
6	(2.42, -1.2)	(-0.18, ...	5.34
7	(2.72, -1.92)	(0.14, 3...	6.42
8	(3.02, -2.64)	(0.46, 4...	7.55
9	(3.32, -3.36)	(0.78, 4...	8.7
10	(3.62, -4.08)	(1.1, 5....	9.87

relative reference versus absolute reference



	A	B	C
1	(1, 2)	(2.7, -0.52)	2.92
2	(1.5, 2.42)	(3, -0.12)	2.73
3	(2, 2.84)	(3.3, 0.28)	[illegible]
4	(2.5, 3.26)	(3.6, 0.68)	2.8
5	(3, 3.68)	(3.9, 1.08)	3.06
6	(3.5, 4.1)	(4.2, 1.48)	3.42
7	(4, 4.52)	(4.5, 1.88)	3.86
8	(4.5, 4.94)	(4.8, 2.28)	4.35
9	(5, 5.36)	(5.1, 2.68)	4.88
10	(5.5, 5.78)	(5.4, 3.08)	5.44

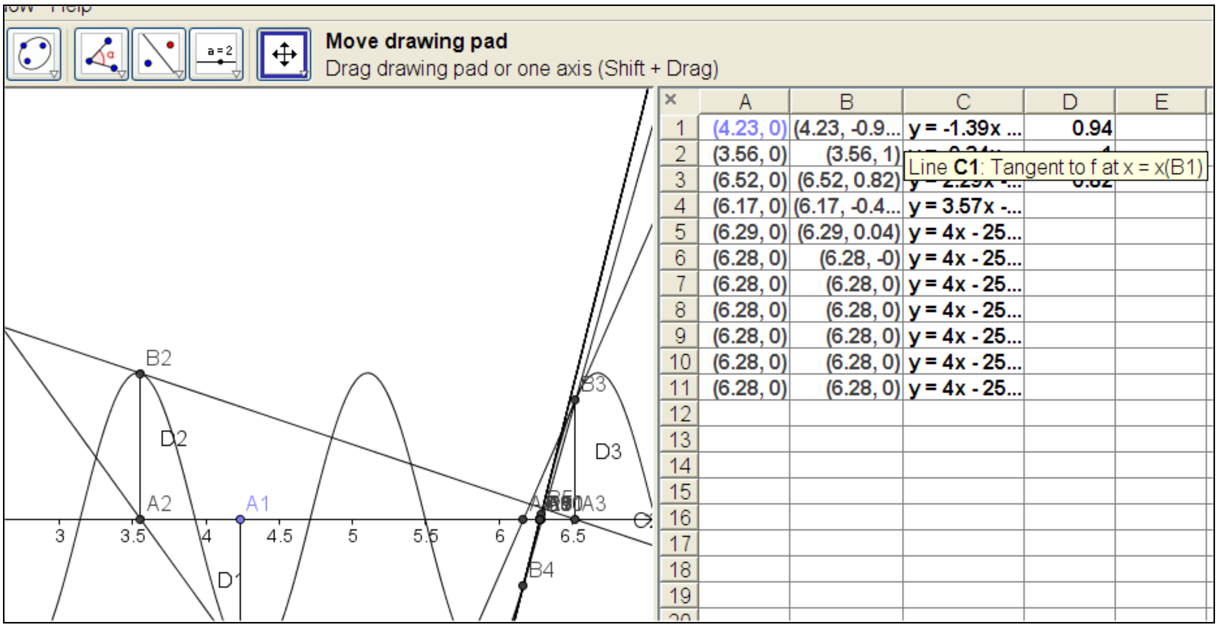
My final example is again from Andreas Lindner:

He uses the spreadsheet for demonstrating the Newton-Raphson algorithm for finding approximatively zeros of a function.

Andreas showed me this file taking a cubic and a quartic as functions. I will take a trig function, because there are many zeros to be expected.

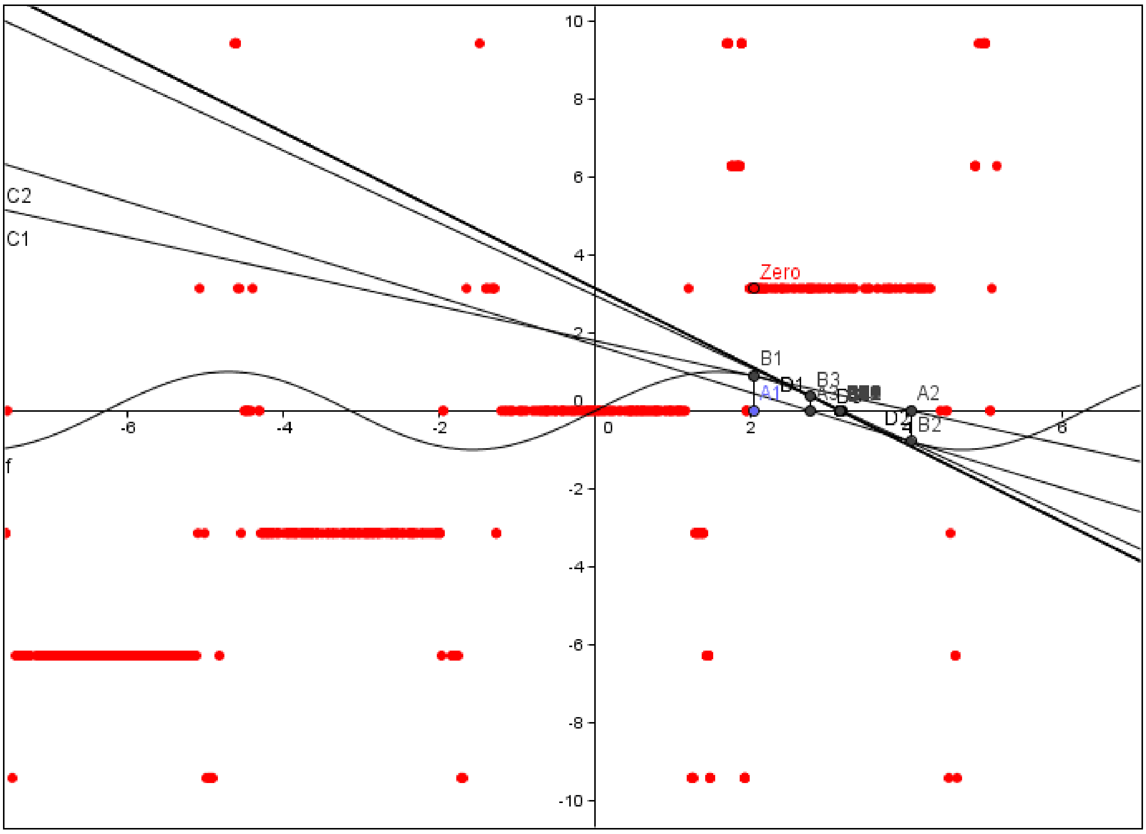
And I added something which I did many years ago with DERIVE. I want to make clear that the choice of the initial value is of high importance for finding the desired zero. Tiny changes in the initial value can have enormous effect and we can observe something like a chaotic and fractal behaviour in the pattern of the limits of the procedure.

See first the Newton-Raphson. The spreadsheet contains the well known formulae in the table.



We move point A ($a|0$) and see where the procedure converges. In the picture above the zero seems to be 2π. But as you can imagine, a tiny move to the right or to the left will end in another zero. My idea was to add the point ($a|\text{zero}(a)$) = (x(A1)$|x$(A11)). A1 and A11 are the cells in the spreadsheet.

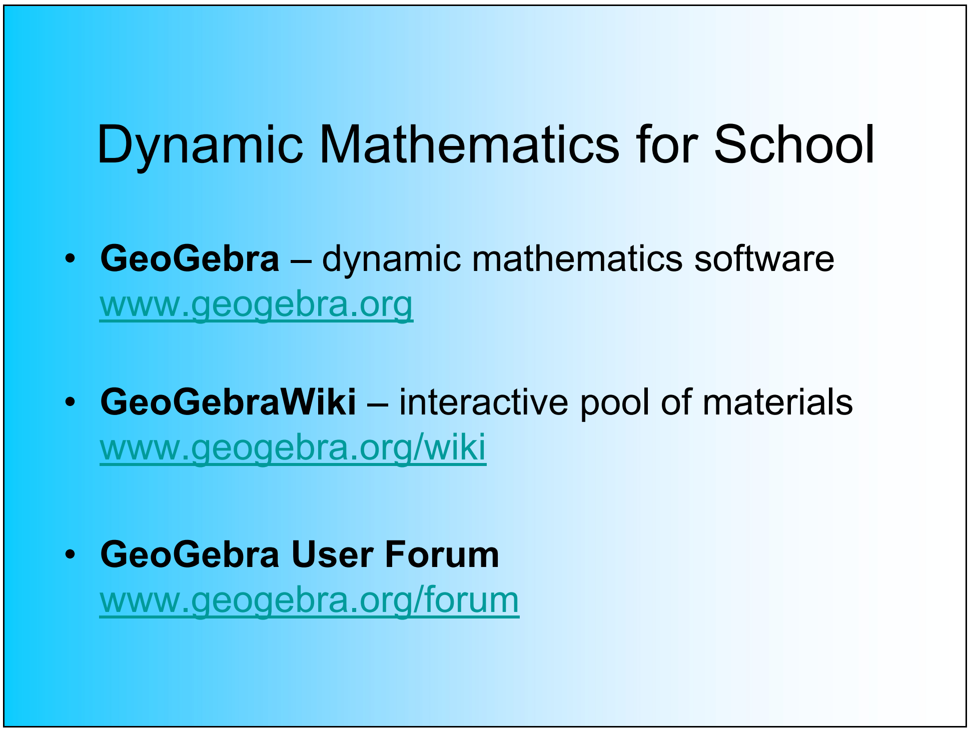
These points appear in red:



The interesting thing is: Zooming in into certain intervals, eg (-5,-4) and zooming in again and again delivers similar patterns. (I can provide an article about this phenomenon on request.)

We will see what future will bring. If it will come true and we will have a tool combining Dynamic Geometry, Computer Algebra, Spreadsheet and possibly 3D Dynamic Geometry and if this tool will keep its simplicity and user friendliness then we can expect a universal tool for math education.

On behalf of the GoeGebra crew I´d like to invite you to visit the following websites.



Thanks for the attention!