Solids of Revolution – from the Integration of a given Function to the Modelling of a Problem with the help of CAS and GeoGebra

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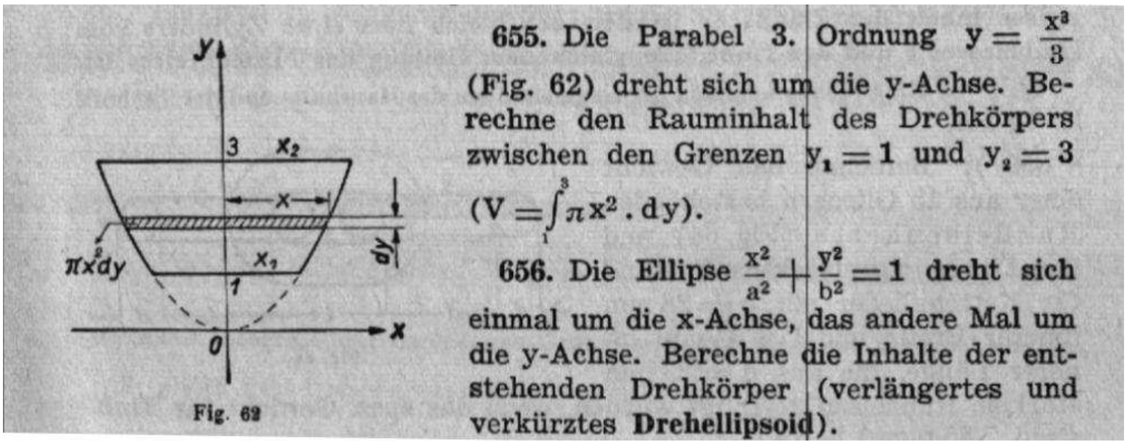
Abstract

After the students in high school have learned to integrate a function, the calculation of the volume of a solid of revolution, like a rotated parabola, is taken as a good applied example. The next step is to calculate the volume of an object of reality which is interpreted as a solid of revolution of a given function f(x). The students do all these calculations in the same way and get the same result. Consequently the teachers can easily decide if a result is right or wrong. If the students have learned to work with a graphical or CAS calculator, they can calculate the volume of solids of revolution in reality by modelling a possible fitted function f(x). Every student has to decide which points of the curve that generates the solid of revolution can be taken and which function will suitably fit the curve. In Austrian high schools teachers use GeoGebra as a software which allows you to insert photographs or scanned material in the geometric window as a background picture. In this case the student and the teacher can control if the graph of the calculated function will fit the generating curve in a useful way.

Introduction

After the 2nd world war the syllabus in Mathematics only recommended an introduction of calculus in form 11 and 12. This meant that only the power function had to be integrated. The teachers had to find applied problems. The students had to calculate areas and volumes of solids of revolution. The following problem 665 with Fig. 62 is typical for the next thirty years.(Rosenberg et al., 1974)



655. Die Parabel 3. Ordnung $y = \frac{x^3}{3}$ (Fig. 62) dreht sich um die y-Achse. Berechne den Rauminhalt des Drehkörpers zwischen den Grenzen $y_1 = 1$ und $y_2 = 3$ $(V = \int_1^3 \pi x^2 . dy)$.

656. Die Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ dreht sich einmal um die x-Achse, das andere Mal um die y-Achse. Berechne die Inhalte der entstehenden Drehkörper (verlängertes und verkürztes **Drehellipsoid**).

$y = \frac{x^3}{3} \rightarrow x^2 = \sqrt[3]{9y^2}$ $\quad V = \int_1^3 \left(\pi.\sqrt[3]{9}.y^{\frac{2}{3}}\right).dx$ $\quad$ Result in the book: $\frac{3\pi}{5}\left(9.\sqrt[3]{3} - \sqrt[3]{9}\right)$

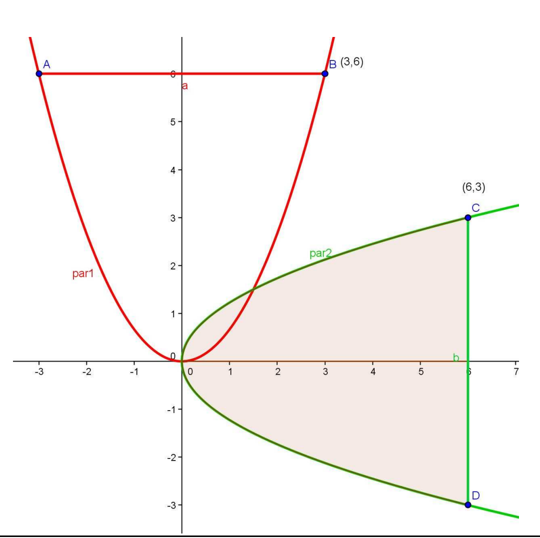
This was a typical result for that time as it was done without a calculator. At the end of the seventies the students were allowed to use calculators and the result was 20.546.

In course of time the students asked for problems which were more realistic. Problem 1162, taken from Szirucsek (1999), is an example for the move in the right direction.

1162 Der Hohlraum eines Weinglases ist im Wesentlichen ein Rotationsparaboloid. Das Glas ist 6 cm tief und hat 6 cm Öffnungsdurchmesser. Wie viel Wein ist enthalten, wenn das Glas gestrichen voll ist bzw. wenn es bis zur halben Höhe gefüllt ist? In welcher Höhe müsste die Markierung für $\frac{1}{16}$ Liter angebracht werden?

If students solve this problem, they have to choose between two formulas they have learned. They can take $x^2 = 2py$ (symmetrical to the y-axe) or $y^2 = 2px$ (symmetrical to the x-axe). If they substitute x=3 and y=6 in $x^2 = 2py$ and x=6 and y=3 in $y^2 = 2px$ they get the equations for the two parabolas par1 and par2:

par1: $x^2 = (3/2)*y \rightarrow y = \frac{2}{3} * x^2$ par2 : $y^2 = (3/2)*x$



Rotation round the y – axe :

$$V = \pi \cdot \int_0^6 x^2 \cdot dy = \pi \cdot \int_0^6 \frac{3}{2} \cdot y \cdot dy$$

$$V = \frac{3\pi}{2} \cdot \frac{y^2}{2} = 27\pi = 84.8$$

Rotation round the x – axe :

$$V = \pi \cdot \int_0^6 y^2 \cdot dx = \pi \cdot \int_0^6 \frac{3}{2} \cdot x \cdot dy$$

$$V = \frac{3\pi}{2} \cdot \frac{x^2}{2} = 27\pi = 84.8$$

To get these results, the students have to choose between two ways. In my tests I often asked the students for two ways of solution. It was a great advantage for the students that they could hope their solution was right if both ways led to the same result. But ex. 1162 is not really an applied problem. The numbers are chosen to allow easy calculating and the position of the coordinate system is fixed with the two formulas they have learned.

New possibilities with the introduction of a CAS, e.g. DERIVE

In Austria a great step in the right direction was taken with the introduction of a CAS in Mathematical Education. In the early 1990s Austria was the first country to buy a general licence for the use of DERIVE in all its grammar schools. One of the advantages was that the students could very quickly create tables and graphic representations of functions. Vice versa the students were enabled to find a function to a given table with the command *FIT(v, A)*. “The elements of the label vector v are the data variables followed by the parameterized expression.” A is the data matrix with the set of points. “When the number of data matrix rows equals the number of parametric variables, *FIT* returns an expression that exactly fits the data, to within roundoff error.”

"When the number of data matrix rows exceeds the number of parametric variables, *FIT* returns an expression that is a least squares fit to the data." (Help of DERIVE 6)

$$FIT([x, a.x^2 + b.x + c], [-1.5, 0; 0.5, -2; 1.5, -1.5]) \text{ simplifies to } \frac{x^2}{2}-\frac{x}{2}-\frac{15}{8}$$

Later on with DERIVE 6 it was possible to insert a photo, e.g. of a wine glass, in the graphic window with the commands „***Option Display Color Picture***“. After you have suitably positioned the co-ordinate system, you can mark, with the cursor, some points lying on the rim of the wine glass and enter the co-ordinates as a data matrix. Now you have to choose a suitable function f(x). If you interpret the rim of the wine glass as a polynomial function, you can get f(x) with *FIT(x, f(x), A)*. With the command *PLOT*, the graph of the function is quickly plotted in the graphic window and you can control how well the graph of f(x) fits the curve. (Böhm, 2006)

Fig.1: The green curve in the figure is the graph of the polynomial function
$$y = 0.002716x^7 - 0.04097x^6 + 0.2281x^5 - 0.5596x^4 + 0.5593x^3 - 0.3814x^2 + 0.8343x + 3.816$$

Now you can quickly calculate the volume of the solid of revolution round the x-axe:

$$V = \pi \cdot \int_0^{4.727} (y(x))^2 \cdot dx = 255.159$$

New possibilities with the introduction of GeoGebra (version 2)

The first Research Projects in Austria (1992-2004) organized by the **ACDCA** (**A**ustrian **C**entre for **D**idactics of **C**omputer **A**lgebra) had the goal to reform the teaching of Mathematics with the use of CAS. In the last projects (after January 2005) ACDCA and **GEOGEBRA** have been cooperating. The name of this software is taken from **GEO**METRY and AL**GEBRA,** because the great advantage of GeoGebra is that an expression in the algebra window corresponds to an object in the geometry window and vice versa. You can simply insert photographs or scanned material in GeoGebra in the geometry window as a **background picture** (Hohenwarter, 2008). Then you can take the coordinates of important points of a curve.

With the command ***Conic through five points*** you can find the curve of the conic and the equation of the conic through the chosen five points simultaneously.

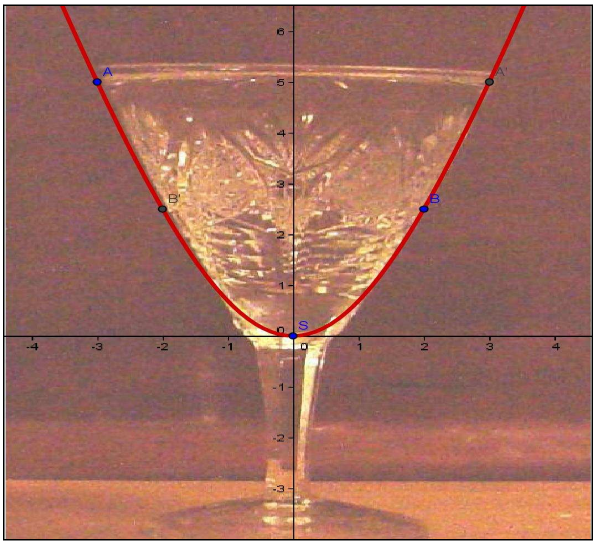


Fig.2: My own photo of a wine glass
hyp : $1875x^2 - 150y^2 - 2625y = 0$

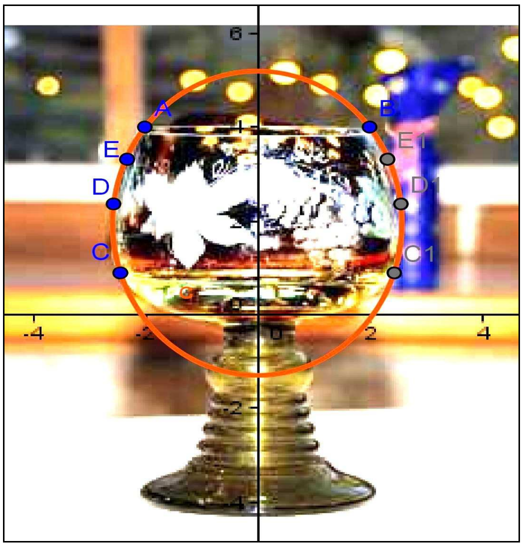


Fig.3: Photo of a wine glass (J. Böhm)
ell : $453.25x^2+287.07y^2-1119.7y = 1927.36$

If you have the goal to find a function to a curve, you have two possibilities:

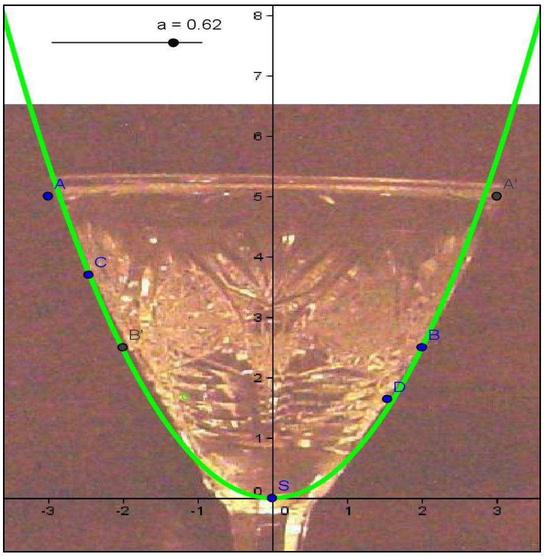


Fig.4: parabola y=a.x² with a=0.62

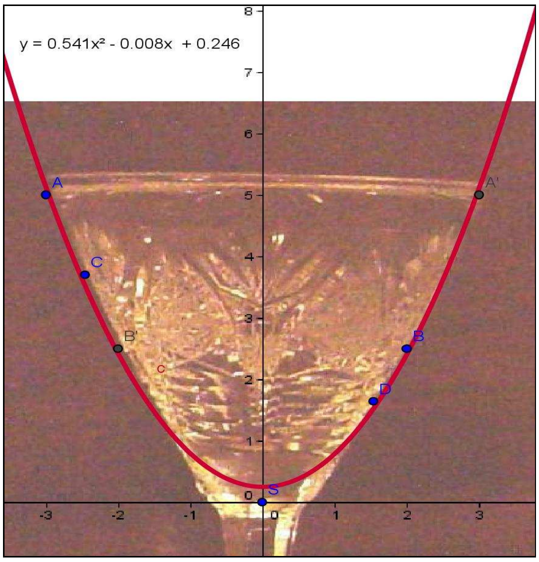


Fig.5: parabola with quadratic regression

In **Fig.2** the curve seems to be a parabola **$y = a.x^2$**, a **parameterized expression** of a function. You take the parameter **a** as slider and move it until the parabola will sufficiently fit the curve. The result for $a = 0.62$ is shown in Fig.4.

In **Fig.5** some points on the rim are marked. If you want to find a function through these points, you have to take a CAS. With TI-Nspire CAS you calculate with the command ***quadratic regression*** the parabola: **$y = 0.541x^2 + 0.008x + 0.246$**

The volume of the solid of revolution round the y-axe with GeoGebra:

In **Fig.4** it is easy to calculate: $V = \pi g \int_0^{y(A)} x^2 g dy = \pi g \int_0^5 \frac{y}{0.62} .dy = 63.34$

In **Fig.5** you need a CAS. First you have to solve **y = 0.541x² + 0.008x + 0.246** to x = f(y), then to compute x² =(f(y))² and to integrate. I did it with DERIVE 6 and TI-Nspire CAS and got the following result:

$$V = \pi g \int_{0.246}^{y(A)} x^2 g dy = \pi g \int_{0.246}^{5} \left(\frac{2.(\sqrt{5}.\sqrt{54100y - 13307} - 2\sqrt{2})^2}{292681}\right).dy = 65.186$$

GeoGebra can integrate, too. "*Integral[Function, Number a, Number b]:* Returns the definite integral of function in the interval *[a , b]*. Note: This command also draws the area between the function graph of *f* and the *x*-axis." (Geogebra Help 3.2)

In **Fig.2** Geogebra calculated the hyperbola: 1875x² - 150y² - 2625y = 0. If you transform to x², you get $g(y) = \frac{\pi g(150gy^2 + 2625y)}{1875}$. In Geogebra you have to enter x instead of y!

$$g(x) = \frac{\pi g(150gx^2 + 2625x)}{1875} \rightarrow V = Integral[g(x), 0, 5] = 65.45$$

New possibilities with the introduction of GeoGebra Pre-Release

If you look into the homepage **www.geogebra.at** , you will read that "GeoGebra is a free and multi-platform dynamic mathematics software for learning and teaching. It has received several educational software awards in Europe and the USA". If you use the order "*Start GeoGebra*" you load the last official version of GeoGebra. If you click "*Future*", you will get to **GeoGebra Pre-Release** and you can try out the currently developed version of GeoGebra. If you click "*help*" and "*about*" you will find the date of the latest version, often the date of the actual day.
I will now treat the wine glass of Fig.1 with **GeoGebra Pre-Release** from April 7, 2009.

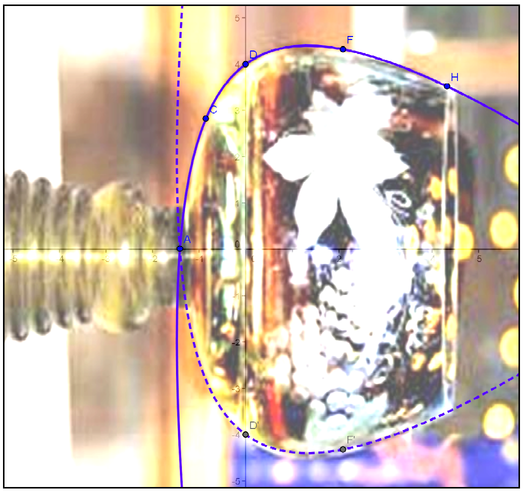


Fig.6: *Conic[H,F,D,C,A]* → hyperbola

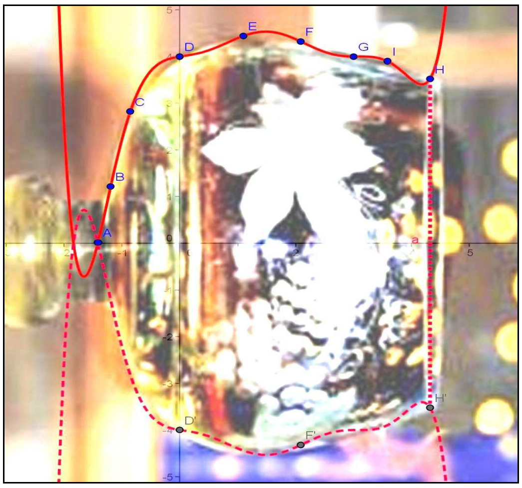


Fig.7: *Polynomial[A,B,C,D,E,F,G,H,I]*

The volume of the solid of revolution round the x-axe with GeoGebra:

In **Fig. 7** the new command "*Polynomial[A, B, C]* creates an interpolation polynomial of degree n-1 through n given points" (GeoGebra 3.0 release notes), with nine points:

$$y = 0.004x^8 - 0.042x^7 + 0.153x^6 - 0.063x^5 - 0.656x^4 + 0.996x^3 - 0.191x^2 + 0.195x + 4$$

V = π*Integral[f(x)², 0.5, 4.321]= **254.042**

In **Fig.6** the command *Conic through five points* computes the hyperbola:
$109.6x^2 + 148.79x.y + 21.4y^2 - 945.66x + 303.85y = 1557.83$
with TI-Nspire CAS: $y = 2.63892g(\sqrt{x^2 + 13.4336x + 17.6907} - 1.31736g(x + 2.04214))$.
with GeoGebra: V = π*Integral[f(x)², 0.5, 4.321] = **255.498**

In **Fig.1** the volume has been worked out with DERIVE: V = **255.159**

Summary of my experiences

Many students in grammar school often ask in Mathematics what the use of learning functions and conics really is and where they exist in reality. Whenever the students take their own photos with their digital cameras of objects which can be interpreted as solids of revolution and then insert them in the geometric window as background pictures, they learn to find conics and graphs of functions. Afterwards they can calculate the volume with integration and are content to have solved an applied problem. The solutions of the students are often very surprising for the teachers and sometimes the numerical results are a little bit different.

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