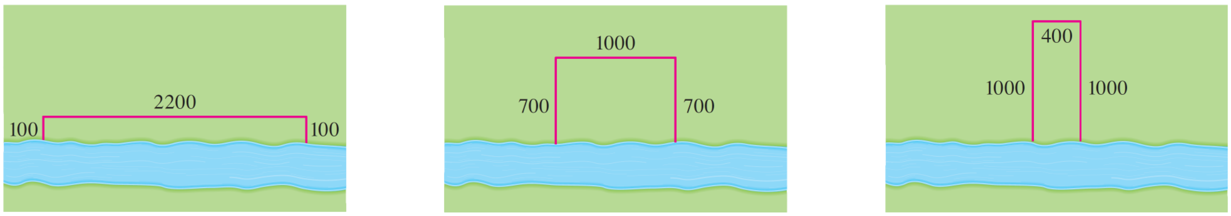
Optimization Problems

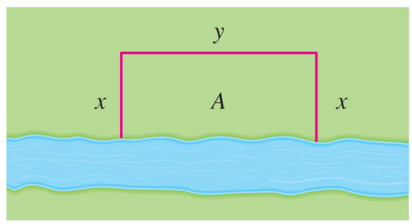
EXAMPLE 1: A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Solution: Note that the area of the field depends on its dimensions:



Area = 100 · 2200 = 220,000 ft^2 Area = 700 · 1000 = 700,000 ft^2 Area = 1000 · 400 = 400,000 ft^2

To solve the problem, we first draw a picture that illustrates the general case:



The next step is to create a corresponding mathematical model:

$$\text{Maximize: } A = xy$$

$$\text{Constraint: } 2x + y = 2400$$

We now solve the second equation for y and substitute the result into the first equation to express A as a function of one variable:

$$2x + y = 2400 \quad \Longrightarrow \quad y = 2400 - 2x \quad \Longrightarrow \quad A = xy = x(2400 - 2x) = 2400x - 2x^2$$

To find the absolute maximum value of $A = 2400x - 2x^2$, we use

THE CLOSED INTERVAL METHOD: To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Step 1 and 2 is the absolute maximum value; the smallest value of these values is the absolute minimum value.

We first note that $0 \le x \le 1200$. The derivative of $A(x)$ is $A'(x) = (2400x - 2x^2)' = 2400 - 4x$, so to find the critical numbers we solve the equation

$$2400 - 4x = 0 \quad \Longrightarrow \quad 2400 = 4x \quad \Longrightarrow \quad x = \frac{2400}{4} = 600$$

To find the maximum value of $A(x)$ we evaluate it at the end points and critical number:

$$A(0) = 0, \qquad A(600) = 2400 \cdot 600 - 2 \cdot 600^2 = 720,000, \qquad A(1200) = 0$$

The Closed Interval Method gives the maximum value as $A(600) = 720,000$ ft^2 and the dimensions are $x = 600$ ft, $y = 2400 - 2 \cdot 600 = 1200$ ft.

EXAMPLE 2: We need to enclose a field with a rectangular fence. We have 500 ft of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.

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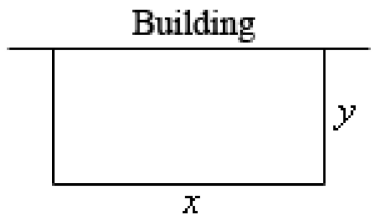


The next step is to create a corresponding mathematical model:

$$\text{Maximize: } A = xy$$
$$\text{Constraint: } x + 2y = 500$$

We now solve the second equation for x and substitute the result into the first equation to express A as a function of one variable:

$$x + 2y = 500 \quad \Longrightarrow \quad x = 500 - 2y \quad \Longrightarrow \quad A = xy = (500 - 2y)y = 500y - 2y^2$$

To find the absolute maximum value of $A = 500y - 2y^2$, we use the Closed Interval Method. We first note that $0 \le y \le 250$. The derivative of $A(y)$ is

$$A'(y) = (500y - 2y^2)' = 500y' - 2(y^2)' = 500 - 4y$$

so to find the critical numbers we solve the equation

$$500 - 4y = 0 \quad \Longrightarrow \quad 500 = 4y \quad \Longrightarrow \quad y = \frac{500}{4} = 125$$

To find the maximum value of $A(y)$ we evaluate it at the end points and critical number:

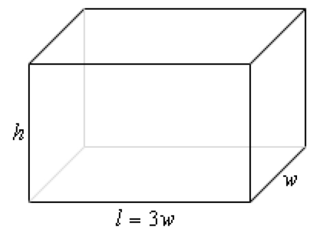
$$A(0) = 0, \qquad A(125) = 500 \cdot 125 - 2 \cdot 125^2 = 31,250, \qquad A(250) = 0$$

The Closed Interval Method gives the maximum value as $A(125) = 31,250$ ft^2 and the dimensions are $y = 125$ ft, $x = 500 - 2 \cdot 125 = 250$ ft.

EXAMPLE 3: We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10/ft^2 and the material used to build the sides cost \$6/ft^2. If the box must have a volume of 50 ft^3 determine the dimensions that will minimize the cost to build the box.

EXAMPLE 3: We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost $\$10/\text{ft}^2$ and the material used to build the sides cost $\$6/\text{ft}^2$. If the box must have a volume of 50 ft^3 determine the dimensions that will minimize the cost to build the box.

Solution: We first draw a picture:



The next step is to create a corresponding mathematical model:

Minimize: $C = 10(2lw) + 6(2wh + 2lh) = 10(2 \cdot 3w \cdot w) + 6(2wh + 2 \cdot 3w \cdot h) = 60w^2 + 48wh$

Constraint: $lwh = 3w^2h = 50$

We now solve the second equation for h and substitute the result into the first equation to express C as a function of one variable:

$$3w^2h = 50 \quad \Longrightarrow \quad h = \frac{50}{3w^2} \quad \Longrightarrow \quad C = 60w^2 + 48wh = 60w^2 + 48w \cdot \frac{50}{3w^2} = 60w^2 + \frac{800}{w}$$

Note that we can't use the Closed Interval Method because the domain of $C(w)$ is $(0, \infty)$ which is not a finite interval. Instead, we will use

FIRST DERIVATIVE TEST FOR ABSOLUTE EXTREME VALUES: Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

The derivative of $C(w)$ is

$$C'(w) = \left(60w^2 + \frac{800}{w}\right)' = 120w - \frac{800}{w^2} = \frac{120w^3}{w^2} - \frac{800}{w^2} = \frac{120w^3 - 800}{w^2}$$

Since $w > 0$, the only critical number is $w = \sqrt[3]{\frac{800}{120}} = \sqrt[3]{\frac{20}{3}} \approx 1.8821$. It is easy to see that $C'(w) < 0$ for all $0 < w < \sqrt[3]{\frac{20}{3}}$ and $C'(w) > 0$ for all $w > \sqrt[3]{\frac{20}{3}}$. Therefore the minimum value of the cost must occur at $w = \sqrt[3]{\frac{20}{3}}$. The dimensions are

$$w = \sqrt[3]{\frac{20}{3}} \approx 1.8821 \text{ ft}, \quad l = 3w = 3\sqrt[3]{\frac{20}{3}} \approx 5.6463 \text{ ft}, \quad h = \frac{50}{3w^2} \approx 4.7050 \text{ ft}$$

and the minimum cost is $C\left(\sqrt[3]{\frac{20}{3}}\right) \approx \$637.60$.

EXAMPLE 4: We want to construct a box with a square base and we only have 10 m^2 of material to use in construction of the box. Assuming that all the material is used in the construction process determine the maximum volume that the box can have.

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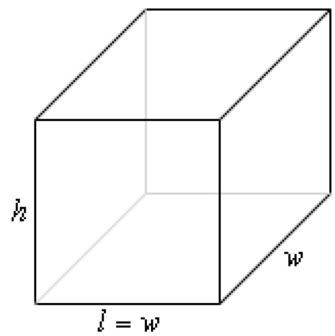

The next step is to create a corresponding mathematical model:

$$\text{Maximize: } V = lwh = w^2h$$

$$\text{Constraint: } 2lw + 2wh + 2lh = 2w^2 + 4wh = 10$$

We now solve the second equation for h and substitute the result into the first equation to express V as a function of one variable:

$$2w^2 + 4wh = 10 \implies h = \frac{10 - 2w^2}{4w} = \frac{5 - w^2}{2w} \implies V = w^2h = w^2\left(\frac{5 - w^2}{2w}\right) = \frac{1}{2}(5w - w^3)$$

Since $w > 0$, we can use only the First Derivative Test for Absolute Extreme Values. The derivative of $V(w)$ is

$$V'(w) = \left(\frac{1}{2}(5w - w^3)\right)' = \frac{1}{2}\left(5w - w^3\right)' = \frac{1}{2}(5 - 3w^2)$$

Since $w > 0$, the only critical number is $w = \sqrt{\frac{5}{3}}$. It is easy to see that $V'(w) > 0$ for all $0 < w < \sqrt{\frac{5}{3}}$ and $V'(w) < 0$ for all $w > \sqrt{\frac{5}{3}}$. Therefore the maximum value of the volume must occur at $w = \sqrt{\frac{5}{3}}$. Finally, the dimensions of the box are

$$w = l = \sqrt{\frac{5}{3}} \approx 1.2910 \text{ m}, \quad h = \frac{5 - w^2}{2w} \approx 1.2910 \text{ m}$$

which means the box with the maximum volume $V = \left(\sqrt{\frac{5}{3}}\right)^3 \approx 2.1517 \text{ m}^3$ is a cube.

EXAMPLE 5: A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.

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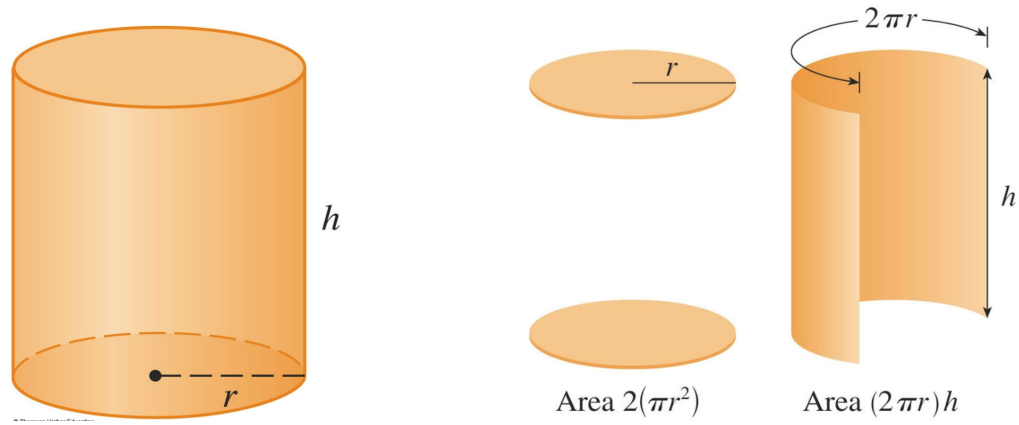
Solution: We first draw a picture:



The next step is to create a corresponding mathematical model:

$$\text{Minimize: } A = 2\pi r^2 + 2\pi rh$$
$$\text{Constraint: } \pi r^2 h = 1500$$

We now solve the second equation for h and substitute the result into the first equation to express A as a function of one variable:

$$\pi r^2 h = 1500 \quad \Longrightarrow \quad h = \frac{1500}{\pi r^2} \quad \Longrightarrow \quad A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \cdot \frac{1500}{\pi r^2} = 2\pi r^2 + \frac{3000}{r}$$

To find the absolute minimum value of $A = 2\pi r^2 + \dfrac{3000}{r}$, we use the First Derivative Test for Absolute Extreme Values. The derivative of $A(r)$ is

$$A'(r) = \left(2\pi r^2 + \frac{3000}{r}\right)' = 4\pi r - \frac{3000}{r^2} = \frac{4\pi r^3}{r^2} - \frac{3000}{r^2} = \frac{4\pi r^3 - 3000}{r^2}$$

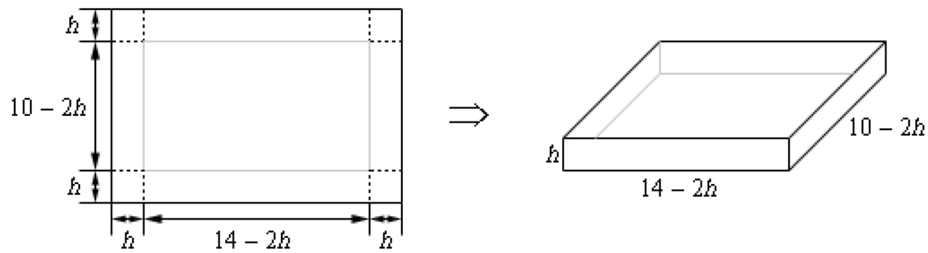
Since $r > 0$, the only critical number is $r = \sqrt[3]{\dfrac{3000}{4\pi}} = \sqrt[3]{\dfrac{750}{\pi}}$. It is easy to see that $A'(r) < 0$ for all $0 < r < \sqrt[3]{\dfrac{750}{\pi}}$ and $A'(r) > 0$ for all $r > \sqrt[3]{\dfrac{750}{\pi}}$. Therefore the minimum value of the area must occur at $r = \sqrt[3]{\dfrac{750}{\pi}} \approx 6.2035$ cm and this value is

$$A\left(\sqrt[3]{\frac{750}{\pi}}\right) \approx 725.3964 \text{ cm}^2$$

Finally, the height of the can is

$$h = \frac{1500}{\pi r^2} = \frac{1500}{\pi(750/\pi)^{2/3}} = 2r \approx 12.4070 \text{ cm}$$

EXAMPLE 6: We have a piece of cardboard that is 14 in by 10 in and we're going to cut out the corners as shown below and fold up the sides to form a box, also shown below. Determine the height of the box that will give a maximum volume.



Solution: We create a corresponding mathematical model:

$$\text{Maximize: } V = h(14-2h)(10-2h) = 140h - 48h^2 + 4h^3$$

It is easy to see that $0 \le h \le 5$. Therefore we can use either the Closed Interval Method or the First Derivative Test for Absolute Extreme Values to find the absolute maximum value of $V = 140h - 48h^2 + 4h^3$.

Closed Interval Method: The derivative of $V(h)$ is

$$V'(h) = (140h - 48h^2 + 4h^3)' = 140 - 96h + 12h^2$$

so to find the critical numbers we solve the equation

$$140 - 96h + 12h^2 = 0 \implies h = \frac{-(-96) \pm \sqrt{(-96)^2 - 4 \cdot 12 \cdot 140}}{2 \cdot 12} = \frac{12 \pm \sqrt{39}}{3} \approx 1.9183,\ 6.0817$$

Since $0 \le h \le 5$, the only critical number that we must consider is $h = \dfrac{12-\sqrt{39}}{3} \approx 1.9183$. To find the maximum value of $V(h)$ we evaluate it at the end points and critical number:

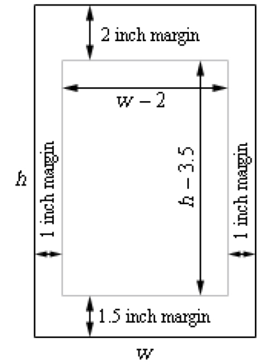
$$V(0) = 0, \qquad V\left(\frac{12-\sqrt{39}}{3}\right) \approx 120.1644, \qquad V(5) = 0$$

Therefore the maximum value of the volume must occur at $h = \dfrac{12-\sqrt{39}}{3} \approx 1.9183$ in and this value is ≈ 120.1644 in^3.

First Derivative Test for Absolute Extreme Values: By the above, $V'(h) = 140 - 96h + 12h^2$ and the only critical number that we must consider is $h = \dfrac{12-\sqrt{39}}{3}$. It is easy to see that $V'(h) > 0$ for all $h < \dfrac{12-\sqrt{39}}{3}$ and $V'(h) < 0$ for all $h > \dfrac{12-\sqrt{39}}{3}$ from $[0,5]$. Therefore the maximum value of the volume must occur at $h = \dfrac{12-\sqrt{39}}{3} \approx 1.9183$ in and this value is $V\left(\dfrac{12-\sqrt{39}}{3}\right) \approx 120.1644$ in^3.

EXAMPLE 7: A printer needs to make a poster that will have a total area of 200 in^2 and will have 1 in margins on the sides, a 2 in margin on the top and a 1.5 in margin on the bottom. What dimensions of the poster will give the largest printed area?

Solution: We first draw a picture. Then we create a corresponding mathematical model:



$$\text{Maximize: } A = (w-2)(h-3.5)$$

$$\text{Constraint: } wh = 200$$

We now solve the second equation for h and substitute the result into the first equation to express A as a function of one variable:

$$wh = 200 \implies h = \frac{200}{w}$$

so

$$A = (w-2)(h-3.5) = (w-2)\left(\frac{200}{w} - 3.5\right) = 207 - 3.5w - \frac{400}{w}$$

It is easy to see that $2 \le w \le \frac{200}{3.5}$. Therefore we can use either the Closed Interval Method or the First Derivative Test for Absolute Extreme Values to find the absolute maximum value of $A = 207 - 3.5w - \dfrac{400}{w}$.

Closed Interval Method: The derivative of $A(w)$ is

$$A'(w) = \left(207 - 3.5w - \frac{400}{w}\right)' = -3.5 + \frac{400}{w^2} = \frac{-3.5w^2}{w^2} + \frac{400}{w^2} = \frac{-3.5w^2 + 400}{w^2}$$

Since $w \ge 2$, the only critical number is $w = \sqrt{\dfrac{400}{3.5}}$. To find the maximum value of $A(w)$ we evaluate it at the end points and critical number:

$$A(2) = 0, \qquad A\left(\sqrt{\frac{400}{3.5}}\right) \approx 120.1644, \qquad A\left(\frac{200}{3.5}\right) = 0$$

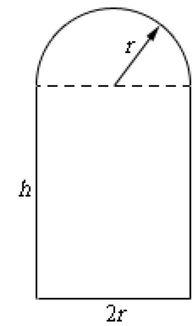
Therefore the maximum value of the area must occur at $w = \sqrt{\dfrac{400}{3.5}} \approx 10.6905$ in and this value is ≈ 132.1669 in^2. Finally, the height of the paper that gives the maximum printed area is

$$h = \frac{200}{w} = \frac{200}{\sqrt{\frac{400}{3.5}}} = 10\sqrt{3.5} \approx 18.7083 \text{ in}$$

First Derivative Test for Absolute Extreme Values: By the above, $A'(w) = \dfrac{-3.5w^2 + 400}{w^2}$ and the only critical number that we must consider is $w = \sqrt{\dfrac{400}{3.5}}$. It is easy to see that $A'(w) > 0$ for all $2 \le w < \sqrt{\dfrac{400}{3.5}}$ and $A'(w) < 0$ for all $w > \sqrt{\dfrac{400}{3.5}}$. Therefore the maximum value of the area must occur at $w = \sqrt{\dfrac{400}{3.5}} \approx 10.6905$ in, $h = 10\sqrt{3.5} \approx 18.7083$ in and this value is ≈ 132.1669 in^2.

EXAMPLE 8: A window is being built and the bottom is a rectangle and the top is a semicircle. If there is 12 m of framing materials what must the dimensions of the window be to let in the most light?

Solution: We first draw a picture. The next step is to create a corresponding mathematical model:



$$\text{Maximize: } A = 2hr + \frac{1}{2}\pi r^2$$

$$\text{Constraint: } 2h + 2r + \pi r = 12$$

We now solve the second equation for h and substitute the result into the first equation to express A as a function of one variable:

$$2h + 2r + \pi r = 12 \quad \Longrightarrow \quad h = 6 - r - \frac{1}{2}\pi r$$

hence

$$A = 2hr + \frac{1}{2}\pi r^2 = 2r\left(6 - r - \frac{1}{2}\pi r\right) + \frac{1}{2}\pi r^2 = 12r - 2r^2 - \frac{1}{2}\pi r^2 = 12r - \left(2 + \frac{1}{2}\pi\right)r^2$$

It is easy to see that $0 \le r \le \dfrac{12}{2+\pi}$. Therefore we can use either the Closed Interval Method or the First Derivative Test for Absolute Extreme Values to find the absolute maximum value of $A = 12r - \left(2 + \dfrac{1}{2}\pi\right)r^2$.

Closed Interval Method: The derivative of $A(r)$ is

$$A'(r) = \left(12r - \left(2 + \frac{1}{2}\pi\right)r^2\right)' = 12 - \left(2 + \frac{1}{2}\pi\right)\cdot 2r = 12 - (4+\pi)r$$

To find the critical numbers we solve the equation

$$12 - (4+\pi)r = 0 \quad \Longrightarrow \quad 12 = (4+\pi)r \quad \Longrightarrow \quad r = \frac{12}{4+\pi}$$

To find the maximum value of $A(r)$ we evaluate it at the end points and critical number:

$$A(0) = 0, \qquad A\left(\frac{12}{4+\pi}\right) \approx 10.0817, \qquad A\left(\frac{12}{2+\pi}\right) = \frac{72\pi}{(2+\pi)^2} \approx 8.5563$$

Therefore the maximum value of the area must occur at $r = \dfrac{12}{4+\pi} \approx 1.6803$ m and this value is $A\left(\dfrac{12}{4+\pi}\right) \approx 10.0817$ m^2. Finally, the height of the window that gives the maximum area is $h = 2r = \dfrac{24}{4+\pi} \approx 3.3606$ m.

First Derivative Test for Absolute Extreme Values: By the above, $A'(r) = 12 - (4+\pi)r$ and the critical number is $r = \dfrac{12}{4+\pi}$. It is easy to see that $A'(r) > 0$ for all $r < \dfrac{12}{4+\pi}$ and $A'(r) < 0$ for all $r > \dfrac{12}{4+\pi}$. Therefore the maximum value of the area must occur at $r = \dfrac{12}{4+\pi} \approx 1.6803$ m and this value is $A\left(\dfrac{12}{4+\pi}\right) \approx 10.0817$ m^2; the height is $h = 2r = \dfrac{24}{4+\pi} \approx 3.3606$ m.

EXAMPLE 9: Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.

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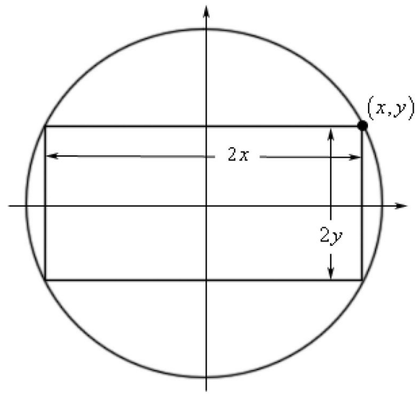
Solution: We first draw a picture:



The next step is to create a corresponding mathematical model:

$$\text{Maximize: } A = 2x \cdot 2y = 4xy$$

$$\text{Constraint: } x^2 + y^2 = 16$$

We can solve the second equation for x and substitute the result into the first equation to express A as a function of one variable. However, this approach involves roots which makes the algebra a bit complicated.

Instead, we square both sides of $A = 4xy$. Note that x and y are both nonnegative. Therefore values that maximize $A = 4xy$ will also maximize $A^2 = 16x^2y^2$ and vice-verse. Putting $B = A^2, u = x^2, v = y^2$, we reformulate our problem in the following way:

$$\text{Maximize: } B = 16uv$$

$$\text{Constraint: } u + v = 16$$

We now solve the second equation for u and substitute the result into the first equation to express B as a function of one variable:

$$u + v = 16 \quad \Longrightarrow \quad u = 16 - v \quad \Longrightarrow \quad B = 16uv = 16(16 - v)v = 256v - 16v^2$$

To find the absolute maximum value of $B = 256v - 16v^2$, we can use either the Closed Interval Method or the First Derivative Test for Absolute Extreme Values. Here we use the First Derivative Test for Absolute Extreme Values. The derivative of $B(v)$ is $B'(v) = 256 - 32v$, so to find the critical numbers we solve the equation

$$256 - 32v = 0 \quad \Longrightarrow \quad 256 = 32v \quad \Longrightarrow \quad v = \frac{256}{32} = 8$$

It is easy to see that $B'(v) > 0$ for all $v < 8$ and $B'(v) < 0$ for all $v > 8$. Therefore the maximum value of the area must occur at $v = 8$ and this area is $A = \sqrt{B} = \sqrt{256 \cdot 8 - 16 \cdot 8^2} = 32$. The dimensions of the rectangle are $y = \sqrt{8} = 2\sqrt{2}$ and $x = \sqrt{u} = \sqrt{16 - v} = \sqrt{16 - 8} = \sqrt{8} = 2\sqrt{2}$. So, this rectangle is a square.

EXAMPLE 10: Determine the points on $y = x^2 + 1$ that are closest to $(0, 2)$.

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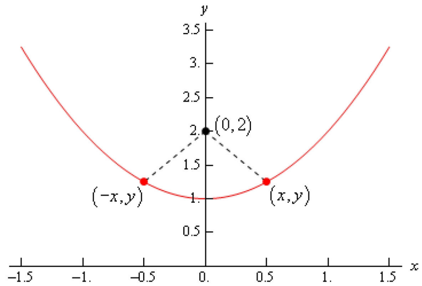


The next step is to create a corresponding mathematical model:

$$\text{Minimize: } d = \sqrt{(x-0)^2 + (y-2)^2} = \sqrt{x^2 + (y-2)^2}$$

$$\text{Constraint: } y = x^2 + 1$$

We can now substitute $y = x^2 + 1$ into the first equation to express d as a function of one variable. However, this approach involves roots which makes the algebra a bit complicated.

Instead, we square both sides of $d = \sqrt{x^2 + (y-2)^2}$. Note that values of x and y that minimize $d = \sqrt{x^2 + (y-2)^2}$ will also minimize $d^2 = x^2 + (y-2)^2$ and vice-verse. Putting $D = d^2$, we can reformulate our problem in the following way:

$$\text{Minimize: } D = x^2 + (y-2)^2$$

$$\text{Constraint: } y = x^2 + 1$$

We now solve the second equation for x^2 and substitute the result into the first equation to express D as a function of one variable:

$$y = x^2 + 1 \quad \Longrightarrow \quad x^2 = y - 1 \quad \Longrightarrow \quad D = x^2 + (y-2)^2 = y - 1 + (y-2)^2 = y^2 - 3y + 3$$

Since there is no upper bound for y, we can use only the First Derivative Test for Absolute Extreme Values to find the absolute minimum value of $D = y^2 - 3y + 3$. The derivative of $D(y)$ is $D'(y) = 2y - 3$, so to find the critical numbers we solve the equation

$$2y - 3 = 0 \quad \Longrightarrow \quad y = \frac{3}{2}$$

It is easy to see that $D'(y) < 0$ for all $y < \dfrac{3}{2}$ and $D'(y) > 0$ for all $y > \dfrac{3}{2}$. Therefore the minimum value of the distance must occur at $y = \dfrac{3}{2}$ and this distance is

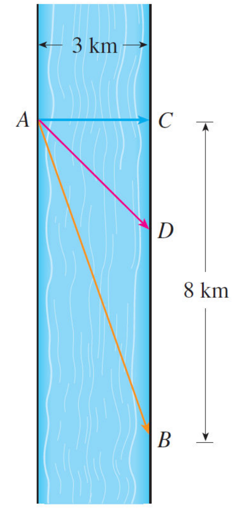
$$d = \sqrt{D} = \sqrt{\left(\frac{3}{2}\right)^2 - 3 \cdot \frac{3}{2} + 3} = \frac{\sqrt{3}}{2}$$

The corresponding x-coordinates are

$$x^2 = y - 1 \quad \Longrightarrow \quad x = \pm\sqrt{y-1} = \pm\sqrt{\frac{3}{2} - 1} = \pm\frac{1}{\sqrt{2}}$$

Thus, the points are $\left(-\dfrac{1}{\sqrt{2}}, \dfrac{3}{2}\right)$ and $\left(\dfrac{1}{\sqrt{2}}, \dfrac{3}{2}\right)$.

EXAMPLE 11: A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could proceed in any of three ways:



1. Row his boat directly across the river to point C and then run to B
2. Row directly to B
3. Row to some point D between C and B and then run to B

If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible?

Solution: If we let x be the distance from C to D, then the running distance is $|DB| = 8 - x$ and the Pythagorean Theorem gives the rowing distance as $|AD| = \sqrt{x^2+9}$. We use the equation

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

Then the rowing time is $\sqrt{x^2+9}/6$ and the running time is $(8-x)/8$, so the total time T as a function of x is

$$T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$$

The domain of this function T is $[0, 8]$. Notice that if $x = 0$ he rows to C and if $x = 8$ he rows directly to B. The derivative of T is

$$\begin{aligned} T'(x) = \left(\frac{(x^2+9)^{1/2}}{6} + \frac{8-x}{8}\right)' &= \frac{1}{6}\left((x^2+9)^{1/2}\right)' + \frac{1}{8}(8-x)' \\ &= \frac{1}{6}\cdot\frac{1}{2}(x^2+9)^{1/2-1}\cdot(x^2+9)' + \frac{1}{8}(8-x)' \\ &= \frac{1}{6}\cdot\frac{1}{2}(x^2+9)^{-1/2}\cdot 2x + \frac{1}{8}\cdot(-1) \\ &= \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8} \end{aligned}$$

Thus, using the fact that $x \geq 0$, we have

$$T'(x) = 0 \;\Leftrightarrow\; \frac{x}{6\sqrt{x^2+9}} = \frac{1}{8} \;\Leftrightarrow\; 4x = 3\sqrt{x^2+9} \;\Leftrightarrow\; 16x^2 = 9(x^2+9) \;\Leftrightarrow\; 7x^2 = 81$$

so the only critical point is $9/\sqrt{7}$. To see whether the minimum occurs at this critical number or at an endpoint of the domain $[0, 8]$, we evaluate T at all three points:

$$T(0) = 1.5 \qquad T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33 \qquad T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

Since the smallest of these values of T occurs when $x = 9/\sqrt{7}$, the absolute minimum value of T must occur there. Thus the man should land the boat at a point $9/\sqrt{7}$ km (≈ 3.4 km) downstream from his starting point.

Applications to Business and Economics

DEFINITION: The **cost function** $C(x)$ is the cost of producing x units of a certain product. The **marginal cost** $C'(x)$ is the rate of change of C with respect to x. The **demand function** (or **price function**) $p(x)$ is the price per unit that the company can charge if it sells x units.

If x units are sold and the price per unit is $p(x)$, then the total revenue is

$$R(x) = xp(x)$$

and R is called the **revenue function**. The derivative R' of the revenue function is called the **marginal revenue function** and is the rate of change of revenue with the respect to the number of units sold. If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

and P is called the **profit function**. The **marginal profit function** is P', the derivative of the profit function.

EXAMPLE: A store has been selling 200 DVD burners a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

Solution: If x is the number of DVD burners sold per week, then the weekly increase in sales is $x - 200$. For each increase of 20 units sold, the price is decreased by \$10. So for each additional unit sold, the decrease in price will be $\frac{1}{20} \times 10$ and the demand function is

$$p(x) = 350 - \frac{10}{20}(x - 200) = 450 - \frac{1}{2}x$$

The revenue function is

$$R(x) = xp(x) = 450x - \frac{1}{2}x^2$$

Since $R'(x) = 450 - x$, we see that $R'(x) = 0$ when $x = 450$. This value of x gives an absolute maximum by the First Derivative Test (or simply by observing that the graph of R is a parabola that opens downward). The corresponding price is

$$p(450) = 450 - \frac{1}{2} \cdot 450 = 225$$

and the rebate is $350 - 225 = 125$. Therefore, to maximize revenue the store should offer a rebate of \$125.